

A SIMPLE CRITERION FOR THE EXISTENCE OF NONREAL EIGENVALUES FOR A CLASS OF 2D AND 3D PAULI OPERATORS

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ABSTRACT. In this work, we investigate the discrete spectrum generated by complex matrix-valued perturbations for a class of 2D and 3D Pauli operators with nonconstant magnetic fields. We establish a simple criterion for the potentials to produce discrete spectrum near the low ground energy of the operators. Moreover, in case of creation of nonreal eigenvalues, this criterion specifies also their location.

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1. INTRODUCTION

1.1. Description of the models. We consider in this article n -dimensional Pauli operators $P_n(b, V)$, $n = 2, 3$, defined as follows. Denote by $X_\perp := (x, y)$ the usual variables on \mathbb{R}^2 and by $X := (X_\perp, X_\parallel)$ those on \mathbb{R}^3 . For $\mathbf{x} = X_\perp \in \mathbb{R}^2$ or $\mathbf{x} = X \in \mathbb{R}^3$, let

$$(1.1) \quad \mathbf{B}(\mathbf{x}) = \begin{cases} b(\mathbf{x}) & \text{for } n = 2, \\ (0, 0, b(\mathbf{x})) & \text{for } n = 3, \end{cases}$$

be a magnetic field such that $b(\mathbf{x}) = b(x, y)$ is an *admissible magnetic field*. Namely, there exists a constant $b_0 > 0$ satisfying

$$(1.2) \quad b(x, y) = b_0 + \tilde{b}(x, y),$$

where \tilde{b} is such that the Poisson equation $\Delta \tilde{\varphi} = \tilde{b}$ admits a solution $\tilde{\varphi} \in C^2(\mathbb{R}^2)$ satisfying $\sup_{(x,y) \in \mathbb{R}^2} |D^\alpha \tilde{\varphi}(x, y)| < \infty$, $\alpha \in \mathbb{Z}_+^2$, $|\alpha| \leq 2$. By defining $\varphi_0(x, y) := \frac{1}{4}b_0(x^2 + y^2)$ on \mathbb{R}^2 and

$$(1.3) \quad \varphi(x, y) := \varphi_0(x, y) + \tilde{\varphi}(x, y),$$

we obtain a magnetic potential $\mathbf{A}_n : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ generating the magnetic field $\mathbf{B} = \text{curl } \mathbf{A}_n$ by setting

$$(1.4) \quad \mathbf{A}_n(\mathbf{x}) = \begin{cases} \mathbf{A}_n(x, y) = (-\partial_y \varphi(x, y), \partial_x \varphi(x, y)) & \text{for } n = 2, \\ \mathbf{A}_n(x, y, X_\parallel) = (-\partial_y \varphi(x, y), \partial_x \varphi(x, y), 0) & \text{for } n = 3. \end{cases}$$

Remark 1.1.

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- (i) The class of admissible magnetic fields described above is essentially the one introduced in [28, 29]. We refer to these papers for more details and examples of admissible magnetic fields.
- (ii) In (1.1), b stands for the intensity of the magnetic field.
- (iii) The case $\tilde{b} = 0$ corresponds to the constant magnetic field of strength $b_0 > 0$.
- (iv) In the three-dimensional case $n = 3$, the magnetic field is of constant direction and points in the X_{\parallel} -direction.

Let $V(\mathbf{x}) = \{V_{\ell k}(\mathbf{x})\}_{\ell, k=1}^2$ be a 2×2 complex matrix-valued potential. Then, the Pauli operators $P_n(b, V)$ acting on $L^2(\mathbb{R}^n) := L^2(\mathbb{R}^n, \mathbb{C}^2)$, $n = 2, 3$, are defined by

$$(1.5) \quad P_n(b, V) := \begin{pmatrix} (-i\nabla - \mathbf{A}_n)^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A}_n)^2 + b \end{pmatrix} + V,$$

initially on $C_0^\infty(\mathbb{R}^n, \mathbb{C}^2)$, and then closed in $L^2(\mathbb{R}^n)$.

For $V = 0$, we have the following result from [28, Propositions 1.1 and 1.2] about the spectrum $\sigma(P_2(b, 0))$ of the operator $P_2(b, 0)$:

Proposition 1.1. *Let b be an admissible magnetic field with $b_0 > 0$. Then, $0 = \inf \sigma(P_2(b, 0))$ is an isolated eigenvalue of infinite multiplicity. More precisely, we have*

$$(1.6) \quad \dim \text{Ker}((-i\nabla - \mathbf{A}_2)^2 - b) = \infty, \quad \dim \text{Ker}((-i\nabla - \mathbf{A}_2)^2 + b) = 0,$$

and

$$(1.7) \quad (0, \zeta) \subset \mathbb{R} \setminus \sigma(P_2(b, 0)),$$

where

$$(1.8) \quad \zeta := 2b_0 e^{-2 \text{osc } \tilde{\varphi}}, \quad \text{osc } \tilde{\varphi} := \sup_{(x, y) \in \mathbb{R}^2} \tilde{\varphi}(x, y) - \inf_{(x, y) \in \mathbb{R}^2} \tilde{\varphi}(x, y).$$

In particular, by [29, Corollary 2.2], we have

$$(1.9) \quad \sigma(P_3(b, 0)) = \sigma_{\text{ac}}(P_3(b, 0)) = [0, \infty).$$

Throughout this paper, our minimal assumptions on the potentials V are the following:

Assumption (A1): For $n = 2$, we assume that

$$(1.10) \quad 0 \not\equiv V_{\ell k}, \quad |V_{\ell k}(x, y)| \leq F(x, y), \quad 1 \leq \ell, k \leq 2,$$

where $F \in (L^{\frac{q}{2}} \cap L^\infty)(\mathbb{R}^2, \mathbb{R}_+^*)$ for some $2 \leq q < \infty$.

Assumption (C1): For $n = 3$, we assume that

$$(1.11) \quad \begin{aligned} & * 0 \not\equiv V_{\ell k}, \quad |V_{\ell k}(x, y, X_{\parallel})| \leq G_{\perp}(x, y) G(X_{\parallel}), \quad 1 \leq \ell, k \leq 2, \\ & * G_{\perp} \in (L^{\frac{q}{2}} \cap L^\infty)(\mathbb{R}^2, \mathbb{R}_+^*) \text{ for some } 2 \leq q < \infty, \\ & * 0 < G(X_{\parallel}) \leq \text{Const.} \langle X_{\parallel} \rangle^{-m}, m > 3, \text{ where } \langle y \rangle := (1 + |y|^2)^{\frac{1}{2}} \text{ for } y \in \mathbb{R}^d. \end{aligned}$$

Examples:

- (i) In Assumptions (A1) and (C1), both F and G_{\perp} can be thought of the function $\mathbb{R}^2 \ni (x, y) \mapsto \langle (x, y) \rangle^{-m_{\perp}}$ with $m_{\perp} > 0$.

- (ii) In Assumption (C1), nonreal-valued potentials V with $V_{\ell k}(x, y, X_{\parallel}) = \mathcal{O}(\langle(x, y, X_{\parallel})\rangle^{-\alpha})$, $1 \leq \ell, k \leq 2$, $\alpha > 3$, can be considered since obviously this implies that

$$V_{\ell k}(x, y, X_{\parallel}) = \mathcal{O}(\langle(x, y)\rangle^{-m_{\perp}} \langle X_{\parallel}\rangle^{-m}), \quad m \in (3, \alpha), \quad m_{\perp} = \alpha - m > 0.$$

1.2. State of the article. Since we will deal with non-self-adjoint operators, for convenience, we introduce some conventional definitions and notations. Let S be a closed operator acting on a separable Hilbert space. An isolated point μ of $\sigma(S)$ lies in $\sigma_{\text{disc}}(S)$, the discrete spectrum of S , if it's algebraic multiplicity

$$(1.12) \quad \text{mult}(\mu) := \text{rank} \left(\frac{1}{2i\pi} \int_{\mathcal{C}} (S - z)^{-1} dz \right)$$

is finite, \mathcal{C} being a small positively oriented circle centred at μ and containing μ as the only point of $\sigma(S)$. Note that the geometric multiplicity of μ , defined by $\dim \text{Ker}(S - \mu)$, satisfies the inequality $\dim \text{Ker}(S - \mu) \leq \text{mult}(\mu)$, equality happening if S is self-adjoint. We define the essential spectrum $\sigma_{\text{ess}}(S)$ of S as the set of points $\mu \in \mathbb{C}$ such that $S - \mu$ is not a Fredholm operator. Under Assumptions (A1) and (C1), we prove that V is relatively compact with respect to $P_n(b, 0)$, $n = 2, 3$. Therefore, due to the Weyl criterion on the invariance of the essential spectrum, we have $\sigma_{\text{ess}}(P_n(b, V)) = \sigma_{\text{ess}}(P_n(b, 0))$, $n = 2, 3$. However, the potential V may generate (complex) discrete spectrum whose only accumulation points are $\sigma_{\text{ess}}(P_n(b, V))$, see [19, Theorem 2.1, p. 373]. The distribution of the discrete spectrum near the essential spectrum for the quantum Hamiltonians has been extensively studied by various authors. However, most of the known results treat the case of self-adjoint electric potentials, see for instance [23, Chap. 11-12], [27, 39, 40, 32, 28, 3, 29, 4] and the references given there. But, recently and during the past years, there has been an increasing interest in the spectral theory of non-self-adjoint differential operators, in particular for the quantum Hamiltonians, see for instance [15, 8, 5, 11, 25, 12, 17, 22, 42, 33, 13, 9, 16]. For a detailed bibliography on the theory, we refer for instance to [42, 9]. Another results on spectral properties for non-self-adjoint operators can be found in Sjöstrand paper [38] and the references given there. Results concerning non-self-adjoint Pauli operators are much more sparse, see for instance [24], where the authors investigated the 1D Pauli equation with complex boundary conditions.

The aim of the present paper is to describe simple methods of obtaining complex eigenvalues asymptotics near the low ground energy 0 of the 2D Pauli operator $P_2(b, V)$, and to show how we can construct complex matrix-valued potentials V generating nonreal eigenvalues near the low ground energy 0 of the 3D Pauli operator $P_3(b, V)$. Our work is closely related to [35, 34], where the author treats the case of the Schrödinger and Dirac operators with constant and nonconstant magnetic fields. Both in these papers and in the present one, the proofs of the results are inspired by previous works (on characteristic values and resonances) for self-adjoint perturbations (see [3, 4]). More precisely, here, in the 2D case, the spectral gap $(0, \zeta)$ in $\sigma(P_2(b, 0))$ allows to reduce the study of $\sigma_{\text{disc}}(P_2(b, V))$ near 0, to that of the zeros of a holomorphic function in a *punctured neighbourhood* of 0 (see Lemma 3.1 and Proposition 3.1). Hence, by this way, we can apply the general approach developed in [4] to solve our problem. On the contrary, in the 3D case, since $\sigma(P_3(b, 0))$ is absolutely continuous, this reduction holds in *half-disks not containing* 0 (see Lemma 6.3 and Proposition 6.2). In that case, [4]'s approach does not work and we have to use the one developed in [3] to solve our problem. The methods of this article also combine functional analysis, complex analysis, functional determinant and spectral properties of Toeplitz operators which appear when making spectral reduction near

the low ground energy 0. The main difficulties come from the matrix-valuedness and the non-selfadjointness of V . Moreover, in the three-dimensional case, unlike the study of the resonances, V is not supposed to be exponentially decreasing with respect to the direction of the magnetic field. Thus, the operator-valued function $z \mapsto \mathcal{T}_V(z)$ defined in Lemma 6.3 is not analytic near the real axis and some limiting absorption principle have to be used (see Proposition 6.2). In contrast with the Laplace operator, in [42], Wang investigated the case of $-\Delta + V$ in $L^2(\mathbb{R}^n)$, $n \geq 2$, V being a dissipative potential, *i.e.* $V(x) = V_1(x) - iV_2(x)$ where V_1 and V_2 are two measurable functions satisfying $V_2(x) \geq 0$, and $V_2(x) > 0$ on an open non empty set. He proved that 0 is not an accumulation point of the complex eigenvalues if the potential decays more rapidly than $|x|^{-2}$. It is still unknown, for more general complex potentials without sign restriction on the imaginary part, whether 0 can be an accumulation point of complex eigenvalues or not. In the present work, we show that in the presence of a magnetic field, the situation is totally different. Even a compactly supported perturbation can produce clusters of eigenvalues near the low ground energy 0 of the operators $P_n(b, V)$, $n = 2, 3$. More precisely, in the case $n = 3$, for some sufficiently small and sufficiently decreasing potential of the form

- ηW , with W a positive Hermitian matrix,
- $\eta \in \mathbb{C}^*$, $\text{Arg}(\eta) \in \pm(\frac{\pi}{2}, \pi)$,

we prove that 0 is an accumulation point of a sequence of eigenvalues which are concentrated along the semi-axis $e^{i(2\text{Arg}(\eta) \mp \pi)}[0, +\infty)$ (see Theorem 2.4). On the contrary, when

- $\text{Arg}(\eta) \in \pm(0, \frac{\pi}{2})$,

this phenomenon disappears (see Corollary 2.1). In the case $n = 2$, the situation is rather different in the sense we prove that 0 is an accumulation point of a sequence of eigenvalues which are concentrated along the semi-axis $\pm e^{i\text{Arg}(\eta)}[0, +\infty)$, for some sufficiently decreasing potential of the form

- ηW , with $\pm W$ a positive Hermitian matrix,
- $\eta \in \mathbb{C}^*$,

(see Theorem 2.2). The case of the Laplace operator is also studied in a recent preprint by Bögli [2], where nonreal potentials decaying at infinity generating infinitely many nonreal eigenvalues accumulating at each point of the essential spectrum $[0, +\infty)$ are constructed. Upper bounds on the number of the complex eigenvalues in small annulus near the low ground energy 0 of the operators $P_n(b, V)$, $n = 2, 3$, are also established here (see Theorems 2.1 and 2.3 respectively).

1.3. Organisation of the paper. The paper is organized as follows. Our main results are stated in Section 2. Section 3 is devoted to the study of the discrete spectrum near the low ground energy for the two-dimensional Pauli operator. The corresponding main results are proved in Sections 4 and 5. Section 6 is devoted to the study of the discrete spectrum near the low ground energy with respect to the three-dimensional Pauli operator, the corresponding main results being proved in Sections 7 and 8. Section 9 is a brief appendix on basic properties of Schatten-von Neumann class ideals, Section 10 a brief appendix on the theory of the index of a finite meromorphic operator-valued function, and Section 11 a brief appendix on the notion of characteristic values of operator-valued functions.

Notations. For a 2×2 matrix $M : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $|M|$ denotes the multiplication operator by the matrix $\sqrt{M^* M}(\mathbf{x}) =: \{|M|_{\ell k}(\mathbf{x})\}$, $1 \leq \ell, k \leq 2$, $\mathbf{x} \in \mathbb{R}^n$, $n = 2, 3$. We will denote $\mathfrak{B}_h(\mathbb{R}^n)$ the set of 2×2 Hermitian matrices on \mathbb{R}^n , $n = 2, 3$. The spectral projection of $L^2(\mathbb{R}^2)$ onto the (infinite-dimensional) kernel of $P_2^- := (-i\partial_x - a_1)^2 + (-i\partial_y - a_2)^2 - b$, will be denoted $p := p(b)$. Here, a_j , $j = 1, 2$, are the components of the magnetic potential \mathbf{A}_2 , so that P_2^- is the first component of the operator $P_2(b, 0)$. The operator $P_2^+ := (-i\partial_x - a_1)^2 + (-i\partial_y - a_2)^2 + b$ will denote his second component.

2. STATEMENT OF THE MAIN RESULTS

This section is devoted to the formulation of our main results. The eigenvalues will be counted according to their algebraic multiplicity defined above. As preparation, we first recall some well-known results on Toeplitz operators. We know from [29, Lemma 2.3] that if $U \in L^q(\mathbb{R}^2)$, $q \geq 1$, then the Toeplitz operator pUp belongs to the Schatten-von Neumann class $\mathcal{S}_q(L^2(\mathbb{R}^2))$ (see Section 9 for the definition of the Schatten classes \mathcal{S}_q). In particular, pUp is a compact operator. Moreover, when it is self-adjoint and positive, the following asymptotics about the quantity $\text{Tr } \mathbf{1}_{(r, \infty)}(pUp)$, $r \searrow 0$, are well-known:

(H1) If $0 \leq U \in C^1(\mathbb{R}^2)$ verifies $U(x, y) = u_0\left(\frac{(x, y)}{\|(x, y)\|}\right) \|(x, y)\|^{-m}(1 + o(1))$, $\|(x, y)\| \rightarrow \infty$, $m > 0$ constant, where u_0 is a non-negative continuous function on \mathbb{S}^1 not vanishing identically, $|\nabla U(x, y)| \leq C_1 \|(x, y)\|^{-m-1}$ with some constant $C_1 > 0$, and if there exists an integrated density of states for the operator P_2^- (see [29] definition (3.11)), then by [29, Lemma 3.3],

$$(2.1) \quad \text{Tr } \mathbf{1}_{(r, \infty)}(pUp) = C_m r^{-2/m}(1 + o(1)), \quad r \searrow 0,$$

where $C_m := \frac{b_0}{4\pi} \int_{\mathbb{S}^1} dt u_0(t)^{2/m}$.

(H2) If $0 \leq U \in L^\infty(\mathbb{R}^2)$ verifies $\ln U(x, y) = -\mu \|(x, y)\|^{2\beta}(1 + o(1))$, $\|(x, y)\| \rightarrow \infty$, with some constants $\beta > 0$ and $\mu > 0$, then by [29, Lemma 3.4],

$$(2.2) \quad \text{Tr } \mathbf{1}_{(r, \infty)}(pUp) = \varphi_\beta(r)(1 + o(1)), \quad r \searrow 0,$$

where for $0 < r < e^{-1}$, we set

$$\varphi_\beta(r) := \begin{cases} \frac{1}{2} b_0 \mu^{-1/\beta} |\ln r|^{1/\beta} & \text{if } 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b_0)} |\ln r| & \text{if } \beta = 1, \\ \frac{\beta}{\beta-1} (\ln |\ln r|)^{-1} |\ln r| & \text{if } 1 < \beta < \infty. \end{cases}$$

(H3) If $0 \leq U \in L^\infty(\mathbb{R}^2)$ has a compact support and if there exists a constant $C > 0$ verifying $C \leq U$ on an open non-empty subset of \mathbb{R}^2 , then by [29, Lemma 3.5],

$$(2.3) \quad \text{Tr } \mathbf{1}_{(r, \infty)}(pUp) = \varphi_\infty(r)(1 + o(1)), \quad r \searrow 0,$$

where $\varphi_\infty(r) := (\ln |\ln r|)^{-1} |\ln r|$, $0 < r < e^{-1}$.

2.1. Results for the case 2D. Let V satisfy Assumption (A1). In that case, we have $|V|_{\ell k}(x, y) = \mathcal{O}(F(x, y))$ for any $1 \leq \ell, k \leq 2$ and any $(x, y) \in \mathbb{R}^2$. Then, [29, Lemma 2.3] implies that the Toeplitz operator $p|V|_{11}p$ belongs to the Schatten-von Neumann class

$\mathcal{S}_q(L^2(\mathbb{R}^2))$, $q \geq 2$. In particular, $p|V|_{11}p$ is a self-adjoint positive compact operator. For ζ defined by (1.8), fix $0 < \epsilon \leq \zeta$ and introduce the punctured disk of 0

$$(2.4) \quad D(0, \epsilon)^* := \{\mu \in \mathbb{C} : 0 < |\mu| < \epsilon\}.$$

Our first main result gives an upper bound on the number of complex eigenvalues near the low ground energy 0 of the operator $P_2(b, 0)$, in small annulus.

Theorem 2.1 (Local upper bound). *Assume that Assumption (A1) holds with $\zeta > \|V\| \ll 1$. Then, there exists $0 < r_0 < \zeta$ such that for any $r > 0$ with $r < r_0 < \frac{3}{2}r$,*

$$(2.5) \quad \#\{\mu \in \sigma_{\text{disc}}(P_2(b, V)) : r < |\mu| < 2r\} \leq C \text{Tr } \mathbf{1}_{(r, \infty)}(p|V|_{11}p) |\ln r| + \mathcal{O}(1),$$

for some constant $C > 0$. In particular, if $|V|_{11}$ is compactly supported, then $\text{Tr } \mathbf{1}_{(r, \infty)}(p|V|_{11}p)$ satisfies the asymptotic (2.3) as $r \searrow 0$.

In order to get asymptotic results, we put some restrictions on the potential V .

Assumption (A2): We assume that

$$(2.6) \quad V = \eta W \text{ with } \eta \in \mathbb{C}^*, \text{ and } W(x, y) = \{W_{\ell k}(x, y)\}_{\ell, k=1}^2 \in \mathfrak{B}_h(\mathbb{R}^2), (x, y) \in \mathbb{R}^2.$$

Denote $J := \text{sign}(W)$ the matrix sign of W satisfying $W = J|W|$. Let P_{\perp} and Q_{\perp} be the orthogonal projections in $L^2(\mathbb{R}^2)$ defined by

$$(2.7) \quad P_{\perp} := \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_{\perp} := I - P_{\perp} = \begin{pmatrix} I - p & 0 \\ 0 & I \end{pmatrix}.$$

For $\mu \in D(0, \epsilon) := D(0, \epsilon)^* \cup \{0\}$, we introduce the operator

$$(2.8) \quad \mathcal{A}_{\perp}(\mu) := J|W|^{\frac{1}{2}}p \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |W|^{\frac{1}{2}} - \mu J|W|^{\frac{1}{2}}(P_2(b, 0) - \mu)^{-1}Q_{\perp}|W|^{\frac{1}{2}}.$$

Note that $\mathcal{A}_{\perp}(\cdot)$ is holomorphic on $D(0, \epsilon)$. Introduce Π the orthogonal projection onto $\text{Ker } \mathcal{A}_{\perp}(0)$ together with the following condition:

$$(2.9) \quad \begin{cases} I - \eta \mathcal{A}'_{\perp}(0)\Pi \text{ is an invertible operator for} \\ W \text{ of definite sign } \pm W(x, y) \geq 0, (x, y) \in \mathbb{R}^2. \end{cases}$$

Remark 2.1.

- (i) There is no loss of generality in saying W is of definite sign $J = \pm$.
- (ii) Actually, $\mathcal{A}'_{\perp}(0)\Pi$ is compact so that condition (2.9) is generically satisfied. Namely, it is fulfilled whenever $\eta \in \mathbb{R} \setminus \{e_n^{-1}, n \in \mathbb{N}\}$, where $(e_n)_n$ denotes the nonzero eigenvalue sequence of the operator $\mathcal{A}'_{\perp}(0)\Pi$.
- (iii) Condition (2.9) is also fulfilled for small potentials V . Namely, for potentials $V = \varepsilon \eta W$ with ε a real number small enough.

If r_0, δ , are two positive fixed constants, and $r > 0$ tending to zero, we define the sector

$$(2.10) \quad \Gamma^{\delta}(r, r_0) := \{x + iy \in \mathbb{C} : r < x < r_0, -\delta x < y < \delta x\}.$$

Theorem 2.2 (Localization, asymptotic behaviours). *Let V satisfy Assumptions (A1) and (A2), with $\eta \in \mathbb{C}^*$ and W of definite sign $J = \pm$. Assume that (2.9) holds. Then, there exists $r_0 > 0$ such that near zero:*

(i) *Localization:* The discrete eigenvalues μ of $P_2(b, \eta W)$ with $0 < |\mu| < r_0$ satisfy

$$(2.11) \quad \mu \in \pm \eta \overline{\Gamma^\delta(r, r_0)},$$

for any $\delta > 0$.

(ii) *Asymptotic:* There exists a sequence $(r_\ell)_\ell$ of positive numbers tending to zero such that

$$(2.12) \quad \#\{\sigma_{\text{disc}}(P_2(b, \eta W)) : r_\ell < |\mu| < r_0\} = \text{Tr} \mathbf{1}_{(r_\ell, \infty)}(p|W|_{11}p)(1 + o(1)),$$

as $\ell \rightarrow \infty$.

(iii) *Asymptotic:* If $|W|_{11}$ satisfies the hypotheses (H1), (H2) and (H3) above, then,

$$(2.13) \quad \#\{\sigma_{\text{disc}}(P_2(b, \eta W)) : r < |\mu| < r_0\} = \text{Tr} \mathbf{1}_{(r, \infty)}(p|W|_{11}p)(1 + o(1)),$$

as $r \searrow 0$.

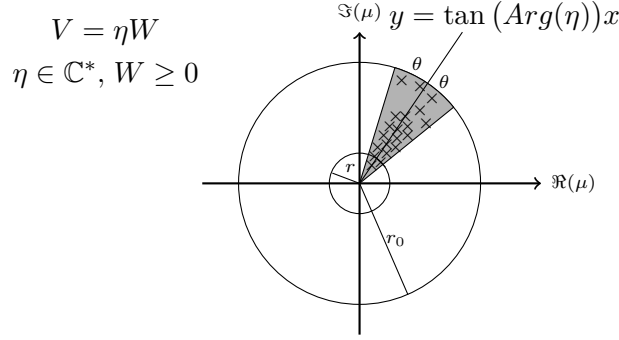


FIGURE 2.1. **A graphic illustration of the localization of the nonreal eigenvalues near zero:** For r_0 small enough, the nonreal eigenvalues μ of $P_2(b, \eta W)$ are localized near the semi-axis $\mu = \eta]0, +\infty)$ in small angular sectors.

Corollaries and Remarks.

- (i) According to (2.1), (2.2) and (2.3), the asymptotics obtained in Theorem 2.2 essentially coincide with those obtained in [28], where the case of the self-adjoint Pauli operator $P_2(b, V)$ with more general assumptions on V is considered. Novelty in this paper is that we consider complex matrix-valued potentials V .
- (ii) Theorem 2.2 is still true if we replace the assumption $J = \pm$ by

$$\text{sign}(W) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This condition is for instance fulfilled by potentials $W \in \mathfrak{B}_h(\mathbb{R}^2, \mathbb{C}^2)$ of the form

$$W = \begin{pmatrix} \pm W_{11} & 0 \\ 0 & W_{22} \end{pmatrix}, \quad W_{11} > 0.$$

- (iii) The existence of nonreal eigenvalues and their accumulation near 0 are ensured by assertion (ii) and (iii) of Theorem 2.2.

- (iv) Another way of stating assertion (i) of Theorem 2.2 is as follows: near the low ground energy of $P_2(b, 0)$, the discrete eigenvalues μ of $P_2(b, \eta W)$ are concentrated around the semi-axis $\pm\eta]0, +\infty)$, respectively for $J = \pm$.
- (v) For $\eta = 1$, we recover the standard self-adjoint electric potentials V of definite sign, and (2.11) becomes $\mu \in \pm]0, +\infty)$. This just amounts to saying that the discrete eigenvalues are localized on the right and the left of 0.
- (vi) If $\eta = i$, we can write (2.11) as $\pm\Im(\mu) \geq 0$ with $|\Re(\mu)| = o(|\mu|)$, see (5.1). This means that the discrete eigenvalues are concentrated in a vicinity of the semi-axis $\pm i]0, +\infty)$.

2.2. Results for the case 3D. Let V satisfy Assumption (C1) and \mathbf{V}_{11} be the multiplication operator by the function (also noted) $\mathbf{V}_{11} : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$(2.14) \quad \mathbf{V}_{11}(x, y) := \frac{1}{2} \int_{\mathbb{R}} dX_{\parallel} |V|_{11}(x, y, X_{\parallel}).$$

Then, as in the case 2D, the Toeplitz operator $p\mathbf{V}_{11}p$ satisfies $p\mathbf{V}_{11}p \in \mathcal{S}_q(L^2(\mathbb{R}^3))$ so that it is a self-adjoint positive compact operator. In the sequel,

$$(2.15) \quad \mathbb{C}_{\pm} := \{z \in \mathbb{C} : \pm\Im(z) > 0\}$$

denotes respectively the upper and lower half-plane. For $0 \leq \varrho_1 < \varrho_2 \leq \zeta$, we introduce the ring

$$(2.16) \quad D(\varrho_1, \varrho_2) := \{z \in \mathbb{C} : \varrho_1 < |z| < \varrho_2\},$$

and the half-rings

$$(2.17) \quad D_{\pm}(\varrho_1, \varrho_2) := D(\varrho_1, \varrho_2) \cap \mathbb{C}_{\pm}.$$

For $\nu > 0$ constant, we define the domains

$$(2.18) \quad D_{\pm}^{\nu}(\varrho_1, \varrho_2) := D_{\pm}(\varrho_1, \varrho_2) \cap \{z \in \mathbb{C} : |\Im(z)| > \nu\}.$$

Our first main result gives upper bounds on the number of complex eigenvalues near the low ground energy 0 of the operator $P_3(b, 0)$, in small half-rings.

Theorem 2.3 (Local upper bounds). *Assume that Assumption (C1) holds with $\zeta > \|V\| \ll 1$. Then, there exists $0 < r_0 < \sqrt{\zeta}$ such that for any $r > 0$ with $r < r_0 < \sqrt{\frac{5}{2}}r$, and any $0 < \nu < 2r^2$,*

$$(2.19) \quad \#\{z \in \sigma_{\text{disc}}(P_3(b, V)) \cap D_{\pm}^{\nu}(r^2, 4r^2)\} \leq C \left(\text{Tr } \mathbf{1}_{(r, \infty)}(p\mathbf{V}_{11}p) |\ln r| \right) + \mathcal{O}(1),$$

for some constant $C > 0$. In particular, if \mathbf{V}_{11} is compactly supported, then $\text{Tr } \mathbf{1}_{(r, \infty)}(p\mathbf{V}_{11}p)$ satisfies the asymptotic (2.3) as $r \searrow 0$.

In what follows below, two kinds of assumption on V are needed in additional.

Assumption (C2): We assume that

$$(2.20) \quad V = \eta W \text{ with } \eta \in \mathbb{C} \setminus \mathbb{R}, \text{ and } W(X) = \{W_{\ell k}(X)\}_{\ell, k=1}^2 \in \mathfrak{B}_h(\mathbb{R}^3), \ X \in \mathbb{R}^3.$$

Remark 2.2.

- (i) In (2.20), when W is of definite sign (i.e. $\pm W \geq 0$), since the change of the sign consists to replace $\text{Arg}(\eta)$ by $\text{Arg}(\eta) + \pi$, then it is sufficient to consider only $W \geq 0$.

- (ii) For $\pm \sin(\text{Arg}(\eta)) > 0$ and $W \geq 0$, the discrete eigenvalues z of the operator $P_3(b, \eta W)$ satisfy $\pm \Im(z) \geq 0$.

With respect to Assumption (C2) above, we introduce \mathbf{W}_{11} , the multiplication operator by the function (also noted) $\mathbf{W}_{11} : \mathbb{R}^2 \rightarrow \mathbb{R}$, defined as in (2.14) with $|V|_{11}$ replaced by $|W|_{11}$. Hence, we introduce the following exponential decay assumption:

Assumption (C3): We assume that the function \mathbf{W}_{11} satisfies

$$(2.21) \quad 0 < \mathbf{W}_{11}(x, y) \leq e^{-C\langle(x,y)\rangle^2}, \quad C > 0.$$

For $\alpha \in \mathbb{R}$ and $\theta > 0$, we introduce the sector

$$(2.22) \quad E_{\pm}(\alpha, \theta) := D(0, \zeta) \setminus \left(e^{i(2\alpha \mp \pi)} e^{i(-2\theta, 2\theta)}(0, \zeta) \right),$$

where we have just excluded around the semi-axis $z = e^{i(2\alpha \mp \pi)}(0, \zeta)$, an angular sector of amplitude 4θ . Hence, we can formulate our second main result in this part as follows:

Theorem 2.4 (Sector free of complex eigenvalues, lower bounds). *Suppose that V satisfies Assumption (C1), and Assumption (C2) with $W \geq 0$, $\pm \text{Arg}(\eta) \in (0, \pi)$. Then, for any $\theta > 0$ small enough, there exists $\varepsilon_0 > 0$ such that:*

- (i) *For any $0 < \varepsilon \leq \varepsilon_0$, the operator $P_3(b, \varepsilon V)$ has no discrete eigenvalues in the sector*

$$(2.23) \quad D_{\pm}(r^2, r_0^2) \cap E_{\pm}(\text{Arg}(\eta), \theta), \quad 0 \leq r < r_0 < \sqrt{\zeta}.$$

- (ii) *If furthermore \mathbf{W}_{11} satisfies Assumption (C3), then for any $0 < \varepsilon \leq \varepsilon_0$, there is an accumulation of nonreal eigenvalues of $P_3(b, \varepsilon V)$ near zero, in a sector around the semi-axis $e^{i(2\text{Arg}(\eta) \mp \pi)}(0, +\infty)$, for*

$$(2.24) \quad \text{Arg}(\eta) \in \pm \left(\frac{\pi}{2}, \pi \right).$$

More precisely, there exists a decreasing sequence of positive numbers $(r_{\ell})_{\ell}$, $r_{\ell} \searrow 0$, satisfying

$$(2.25) \quad \begin{aligned} \# \left\{ z \in \sigma_{\text{disc}}(P_3(b, \varepsilon V)) \cap D_{\pm}(\varepsilon^2 r_{\ell+1}^2, \varepsilon^2 r_{\ell}^2) \cap \left(e^{i(2\text{Arg}(\eta) \mp \pi)} e^{i(-2\theta, 2\theta)}(0, \zeta) \right) \right\} \\ \geq \text{Tr} \mathbf{1}_{(r_{\ell+1}, r_{\ell})}(p \mathbf{W}_{11} p). \end{aligned}$$

A graphic illustration of Theorem 2.4 is given in Figure 2.2.

Remark 2.3.

- (i) Theorem 2.4 and [3, Theorem 2] are quite similar in there structure, although in [3], is considered the 3D Schrödinger operator with constant magnetic field and self-adjoint potentials decaying exponentially in the direction of the magnetic field, to study the resonances near the Landau levels.
- (ii) If we set $\delta := \tan(\theta)$, then the proof of Theorem 2.4 (see (8.7) and (8.16)) shows that for $0 < r_0 < \sqrt{\zeta}$, the parameter $\varepsilon_0 > 0$ above depends on θ as follows:

$$(2.26) \quad \varepsilon_0 < \left(C \sqrt{1 + \delta^{-2}} \right)^{-1} \min \left(1, C_1(\delta, \nu)^{-1} e^{-\Gamma_q(C_1(\delta, \nu) + 1)^{\lceil q \rceil}} \right),$$

$$\begin{aligned} Arg(\eta) &\in (\frac{\pi}{2}, \pi), W \geq 0 \\ V &= \eta W \end{aligned}$$

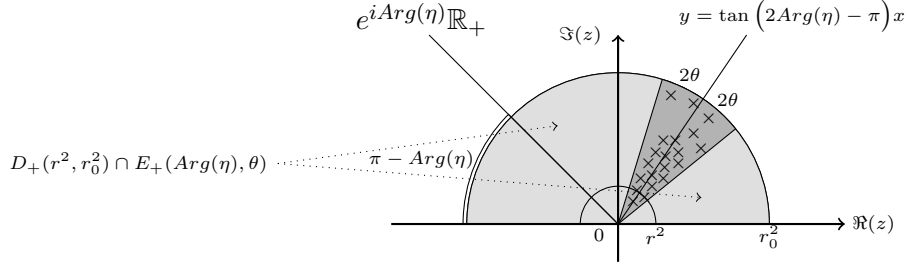


FIGURE 2.2. **A graphic illustration of the localization of the non-real eigenvalues near the low ground energy 0:** For θ small enough and $0 < \varepsilon \leq \varepsilon_0$, $P_3(b, \varepsilon V)$ has no complex eigenvalues in $D_+(r^2, r_0^2) \cap E_+(Arg(\eta), \theta)$ (see (i) of Theorem 2.4). They are localized around the semi-axis $z = e^{i(2Arg(\eta) - \pi)]0, +\infty)$ (see (ii) of Theorem 2.4).

for some uniform constants C , ν positive, where $C_1(\delta, \nu)$ is the constant defined by (8.17), Γ_q by (9.3), and $\lceil q \rceil$ by (9.2).

An immediate consequence of assertion (i) of Theorem 2.4 together with (ii) of Remark 2.2 is the following:

Corollary 2.1 (Non cluster of complex eigenvalues). *Let the assumptions of Theorem 2.4 be fulfilled. Then, for any η satisfying $\text{Arg}(\eta) \in \pm(0, \frac{\pi}{2})$, there is no accumulation of discrete eigenvalues of $P_3(b, \varepsilon V)$ near zero, for $0 < \varepsilon \leq \varepsilon_0$.*

Actually, we expect this to be a general phenomenon in the following sense:

Conjecture 2.1. *Let $V = \eta W$ satisfy Assumption (C1), with $\eta \in \mathbb{C} \setminus \mathbb{R}e^{ik\frac{\pi}{2}}$, $k \in \mathbb{Z}$, and $W \in \mathfrak{B}_h(\mathbb{R}^3)$ of definite sign. Then, there is no accumulation of complex eigenvalues of $P_3(b, V)$ near zero if and only if $\Re(V) > 0$.*

Remark 2.4. To make our results in perspective, let us mention that more general situations such as: more general magnetic field in the 3D case (not supported only in the X_{\parallel} axis), and/or to consider perturbations of the magnetic field itself, are open problems.

3. DISCRETE EIGENVALUES FOR THE 2D PROBLEM

Throughout Sections 3-5, $D(0, \epsilon)^*$ stands for the punctured disk given by (2.4). Here and in the rest of the paper, ζ is the constant defined by (1.8).

3.1. Preliminary results. Let P_\perp and $Q_\perp := I - P_\perp$ be the orthogonal projections defined by (2.7). For $\mu \notin \sigma(P_2(b, 0))$, on account of (1.5) with $n = 2$ and Proposition 1.1, we clearly have

$$(3.1) \quad (P_2(b, 0) - \mu)^{-1} P_{\perp} = -p \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, for any μ lying in the resolvent set of $P_2(b, 0)$, we have

$$(3.2) \quad (P_2(b, 0) - \mu)^{-1} = -p \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 0 \end{pmatrix} + (P_2(b, 0) - \mu)^{-1} Q_{\perp}.$$

We begin with a general result on the first term of the r.h.s. of (3.2).

Lemma 3.1. *Let $U \in L^q(\mathbb{R}^2)$ with $q \in [2, +\infty)$. Then, the operator-valued function*

$$D(0, \epsilon)^* \ni \mu \mapsto U(P_2(b, 0) - \mu)^{-1} P_{\perp}$$

is holomorphic with values in $\mathcal{S}_q(L^2(\mathbb{R}^2))$. Furthermore,

$$(3.3) \quad \left\| U(P_2(b, 0) - \mu)^{-1} P_{\perp} \right\|_{\mathcal{S}_q}^q \leq \frac{b_0 e^{2\text{osc } \tilde{\varphi}}}{2\pi \mu^q} \|U\|_{L^q}^q.$$

Proof. The holomorphicity on $D(0, \epsilon)^*$ is evident. Let us show (3.3).

Thanks to (3.1), we have

$$(3.4) \quad U(P_2(b, 0) - \mu)^{-1} P_{\perp} = -Up \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

As in [29, Proof of Lemma 2.4], it can be proved that $Up \in \mathcal{S}_q(L^2(\mathbb{R}^2))$ with

$$(3.5) \quad \|Up\|_{\mathcal{S}_q}^q \leq \frac{b_0}{2\pi} e^{2\text{osc } \tilde{\varphi}} \|U\|_{L^q}^q.$$

Then, bound (3.3) follows by combining (3.4) and (3.5). \square

The next result concerns the second term of the r.h.s. of (3.2).

Lemma 3.2. *Under the assumptions of Lemma 3.1, the operator-valued function*

$$\mathbb{C} \setminus [\zeta, +\infty) \ni \mu \mapsto U(P_2(b, 0) - \mu)^{-1} Q_{\perp}$$

is holomorphic with values in $\mathcal{S}_q(L^2(\mathbb{R}^2))$. Moreover,

$$(3.6) \quad \left\| U(P_2(b, 0) - \mu)^{-1} Q_{\perp} \right\|_{\mathcal{S}_q}^q \leq C \|U\|_{L^q}^q \left(1 + \frac{|\mu + 1|}{\text{dist}(\mu, [\zeta, +\infty))} \right)^q,$$

where $C = C(q)$ is a constant depending on q .

Proof. For μ lying in the resolvent set of $P_2(b, 0)$, (1.5) implies that we have

$$(3.7) \quad (P_2(b, 0) - \mu)^{-1} Q_{\perp} = \begin{pmatrix} (P_2^- - \mu)^{-1} (I - p) & 0 \\ 0 & (P_2^+ - \mu)^{-1} \end{pmatrix} = (P_2^- - \mu)^{-1} (I - p) \oplus (P_2^+ - \mu)^{-1}.$$

Then, $\mathbb{C} \setminus [\zeta, +\infty) \ni \mu \mapsto (P_2^- - \mu)^{-1} (I - p) \oplus (P_2^+ - \mu)^{-1}$ is well defined and analytic since $\mathbb{C} \setminus [\zeta, +\infty)$ is included in the resolvent set of P_2^- defined on $(I - p) \text{Dom}(P_2^-)$ and P_2^+ defined on $\text{Dom}(P_2^+)$. Thus, $\mu \mapsto U(P_2(b, 0) - \mu)^{-1} Q_{\perp}$ is holomorphic on $\mathbb{C} \setminus [\zeta, +\infty)$.

Now, let us prove estimate (3.6). In what follows below, constants are generic. Namely changing from a relation to another. First, let us show that (3.6) holds if q is even.

Identity (3.7) implies that we have

$$(3.8) \quad \left\| U(P_2(b, 0) - \mu)^{-1} Q_{\perp} \right\|_{\mathcal{S}_q}^q \leq \left\| U(P_2^- - \mu)^{-1} (I - p) \right\|_{\mathcal{S}_q}^q + \left\| U(P_2^+ - \mu)^{-1} \right\|_{\mathcal{S}_q}^q.$$

Let us focus on the first term of the r.h.s. of (3.8). We have

$$(3.9) \quad \left\| U(P_2^- - \mu)^{-1}(I - p) \right\|_{\mathcal{S}_q}^q \leq \left\| U(P_2^- + 1)^{-1} \right\|_{\mathcal{S}_q}^q \left\| (P_2^- + 1)(P_2^- - \mu)^{-1}(I - p) \right\|_{\mathcal{S}_q}^q.$$

By the Spectral mapping theorem, we have

$$(3.10) \quad \left\| (P_2^- + 1)(P_2^- - \mu)^{-1}(I - p) \right\|_{\mathcal{S}_q}^q \leq \sup_{s \in [\zeta, +\infty)}^q \left| \frac{s+1}{s-\mu} \right|.$$

Using the resolvent equation, the boundedness of the magnetic field b , and applying the diamagnetic inequality (see [1, Theorem 2.3] and [37, Theorem 2.13]) which is only valid when q is even, we get

$$(3.11) \quad \begin{aligned} \left\| U(P_2^- + 1)^{-1} \right\|_{\mathcal{S}_q}^q &\leq \left\| I + (P_2^- + 1)^{-1}b \right\|_{\mathcal{S}_q}^q \left\| U((-i\nabla - \mathbf{A})^2 + 1)^{-1} \right\|_{\mathcal{S}_q}^q \\ &\leq C \left\| U(-\Delta + 1)^{-1} \right\|_{\mathcal{S}_q}^q. \end{aligned}$$

By the standard criterion [37, Theorem 4.1], we have

$$(3.12) \quad \left\| U(-\Delta + 1)^{-1} \right\|_{\mathcal{S}_q}^q \leq C \|U\|_{L^q}^q \left\| (|\cdot|^2 + 1)^{-1} \right\|_{L^q}^q.$$

Combining (3.9), (3.10), (3.11) and (3.12), we obtain

$$(3.13) \quad \begin{aligned} \left\| U(P_2^- - \mu)^{-1}(I - p) \right\|_{\mathcal{S}_q}^q &\leq C \|U\|_{L^q}^q \sup_{s \in [\zeta, +\infty)}^q \left| \frac{s+1}{s-\mu} \right| \\ &\leq C \|U\|_{L^q}^q \left(1 + \frac{|\mu+1|}{\text{dist}(\mu, [\zeta, +\infty))} \right)^q. \end{aligned}$$

Arguing similarly, it can be proved that we have

$$(3.14) \quad \left\| U(P_2^+ - \mu)^{-1} \right\|_{\mathcal{S}_q}^q \leq C \|U\|_{L^q}^q \left(1 + \frac{|\mu+1|}{\text{dist}(\mu, [\zeta, +\infty))} \right)^q.$$

Hence, for q even, (3.6) holds by putting together (3.8), (3.13) and (3.14).

To prove that (3.6) holds for any $q \geq 2$, we use interpolation method. If q verifies $q > 2$, then clearly there exists even integers $q_0 < q_1$ satisfying $q \in (q_0, q_1)$ and $q_0 \geq 2$. Consider $\gamma \in (0, 1)$ such that $\frac{1}{q} = \frac{1-\gamma}{q_0} + \frac{\gamma}{q_1}$ and introduce the operator

$$L^{q_i}(\mathbb{R}^2) \ni U \xrightarrow{T} U(P_2(b, 0) - \mu)^{-1} Q_{\perp} \in \mathcal{S}_{q_i}, \quad i = 0, 1.$$

Denote $C_i = C(q_i)$ the constant appearing in (3.6) for $i = 0, 1$, and define

$$C(\mu, q_i) := C_i^{\frac{1}{q_i}} \left(1 + \frac{|\mu+1|}{\text{dist}(\mu, [\zeta, +\infty))} \right).$$

By (3.6), $\|T\| \leq C(\mu, q_i)$ for $i = 0, 1$. Using the Riesz-Thorin Theorem (see for instance [14, Sub. 5 of Chap. 6], [31], [41], [26, Chap. 2]), we interpolate between q_0 and q_1 to obtain the extension $T : L^q(\mathbb{R}^2) \rightarrow \mathcal{S}_q$, with

$$\|T\| \leq C(\mu, q_0)^{1-\gamma} C(\mu, q_1)^{\gamma} \leq C(q)^{\frac{1}{q}} \left(1 + \frac{|\mu+1|}{\text{dist}(\mu, [\zeta, +\infty))} \right).$$

Then, in particular, we have for any $U \in L^q(\mathbb{R}^2)$

$$\|T(U)\|_{\mathcal{S}_q} \leq C(q)^{\frac{1}{q}} \left(1 + \frac{|\mu + 1|}{\text{dist}(\mu, [\zeta, +\infty))} \right) \|U\|_{L^q},$$

or equivalently bound (3.6), and the lemma follows. \square

Now, observe that Assumption (A1) given by (1.10), implies that there exists a bounded operator \mathcal{M} such that $|V|^{\frac{1}{2}} = \mathcal{M}F^{\frac{1}{2}}$ with F satisfying (1.10). Consequently, since the radius ϵ of $D(0, \epsilon)^*$ satisfies $0 < \epsilon < \zeta$ and V is bounded, then by combining identities (3.1) and (3.2) with Lemmas 3.1 and 3.2, we obtain the following:

Lemma 3.3. *The operator-valued function*

$$D(0, \epsilon)^* \ni \mu \longmapsto \mathcal{T}_{V, \perp}(\mu) := \tilde{J}|V|^{\frac{1}{2}}(P_2(b, 0) - \mu)^{-1}|V|^{\frac{1}{2}}$$

is holomorphic with values in $\mathcal{S}_q(L^2(\mathbb{R}^2))$, where \tilde{J} is defined by the polar decomposition $V = \tilde{J}|V|$ of V .

3.2. Reduction of the problem. We reduce the study of the discrete eigenvalues of $P_2(b, V)$ near zero, to that of the zeros of a holomorphic function in a punctured neighbourhood of zero.

In what follows below, the regularized determinant $\det_{[\mathfrak{q}]}(\bullet)$ is defined in Appendix A 1. Similarly to Lemma 3.3, it can be shown that $V(P_2(b, 0) - \cdot)^{-1}$ is holomorphic in $D(0, \epsilon)^*$ with values in $\mathcal{S}_q(L^2(\mathbb{R}^2))$. Then, $\det_{[\mathfrak{q}]}(I + V(P_2(b, 0) - \mu))$ is well defined by (9.2). It is well known, see for instance [37, Chap. 9], that

$$(3.15) \quad \mu \in \sigma_{\text{disc}}(P_2(b, V)) \Leftrightarrow f_q(\mu) := \det_{[\mathfrak{q}]}(I + V(P_2(b, 0) - \mu)^{-1}) = 0.$$

Since $V(P_2(b, 0) - \cdot)^{-1}$ is holomorphic in $D(0, \epsilon)^*$, then so is the function f_q by Property **d**) of Appendix A 1. Moreover, the algebraic multiplicity of μ as a discrete eigenvalue of $P_2(b, V)$ is equal to its order as a zero of f_q .

In the next proposition, the quantity $\text{Ind}_{\mathcal{C}}(\bullet)$ appearing in the r.h.s. of (3.16) is defined in Appendix A 2.

Proposition 3.1. *Let $\mathcal{T}_{V, \perp}(\mu)$ be the operator defined in Lemma 3.3. Then, the following assertions are equivalent:*

- (i) $\mu \in D(0, \epsilon)^*$ is a discrete eigenvalue of $P_2(b, V)$,
- (ii) $\det_{[\mathfrak{q}]}(I + \mathcal{T}_{V, \perp}(\mu)) = 0$,
- (iii) -1 is an eigenvalue of $\mathcal{T}_{V, \perp}(\mu)$.

Moreover,

$$(3.16) \quad \text{mult}(\mu) = \text{Ind}_{\mathcal{C}}(I + \mathcal{T}_{V, \perp}(\cdot)),$$

where \mathcal{C} is a small contour positively oriented, containing μ as the unique discrete eigenvalue of $P_2(b, V)$.

Proof. The equivalence between assertions (i) and (ii) follows obviously from (3.15) and the equality

$$\det_{[q]}(I + V(P_2(b, 0) - \mu)) = \det_{[q]}(I + \tilde{J}|V|^{\frac{1}{2}}(P_2(b, 0) - \mu)^{-1}|V|^{\frac{1}{2}}),$$

thanks to Property **b)** of Appendix A 1.

The equivalence between assertions (ii) and (iii) follows from Property **c)** of Appendix A 1.

We prove now (3.16). To this end, we consider the function f_q defined by (3.15). According to the comment just after (3.15), we have

$$(3.17) \quad \text{mult}(\mu) = \text{ind}_{\mathcal{C}} f_q,$$

where the r.h.s. of (3.17) is the index defined by (10.2) of the holomorphic function f_q with respect to the contour \mathcal{C} . We thus get (3.16) easily from the equality

$$\text{ind}_{\mathcal{C}} f_q = \text{Ind}_{\mathcal{C}}(I + \mathcal{T}_{V,\perp}(\cdot)),$$

(see for instance [4, (2.6)] for more details). □

3.3. Decomposition of the weighted resolvent. Our goal in this section is to split the weighted resolvent $\mathcal{T}_{V,\perp}(\mu) := \tilde{J}|V|^{\frac{1}{2}}(P_2(b, 0) - \mu)^{-1}|V|^{\frac{1}{2}}$ into two parts which are respectively meromorphic and holomorphic in a neighbourhood of zero. The potential V is assumed to satisfy Assumption (A1).

The next proposition is a direct consequence of identities (3.1), (3.2) and Lemma 3.3.

Proposition 3.2. *For $\mu \in D(0, \epsilon)^*$, we have*

$$(3.18) \quad \mathcal{T}_{V,\perp}(\mu) = -\frac{\tilde{J}}{\mu}|V|^{\frac{1}{2}} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}} + \mathcal{A}_{\perp}(\mu),$$

where the operator $\mathcal{A}_{\perp}(\mu) := \tilde{J}|V|^{\frac{1}{2}}(P_2(b, 0) - \mu)^{-1}\mathbf{Q}_{\perp}|V|^{\frac{1}{2}} \in \mathcal{S}_q(L^2(\mathbb{R}^2))$ is holomorphic in the open disk $D(0, \epsilon) := D(0, \epsilon)^* \cup \{0\}$.

Remark 3.1.

- (i) Thanks to Lemma 3.1, $|V|^{\frac{1}{2}} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}}$ is a compact operator. Then, for any $r > 0$, we have

$$(3.19) \quad \text{Tr} \mathbf{1}_{(r,\infty)} \left(|V|^{\frac{1}{2}} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}} \right) = \text{Tr} \mathbf{1}_{(r,\infty)} \left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} |V| \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \right) = \text{Tr} \mathbf{1}_{(r,\infty)} (p|V|_{11}p),$$

where we recall that the $|V|_{\ell k}$, $1 \leq \ell, k \leq 2$, are the coefficients of the matrix $|V|$.

- (ii) If V satisfies Assumption (A2) given by (2.6), then Proposition 3.2 holds with \tilde{J} replaced by $J\eta$, $J := \text{sign}(W)$, and $|V|_{11}$ replaced by $|W|_{11}$ in (3.19).

4. PROOF OF THEOREM 2.1: UPPER BOUND

The proof will be divided into two steps.

4.1. A preliminary result. Let

$$(4.1) \quad N(P_2(b, V)) := \{\langle P_2(b, V)f, f \rangle : f \in \text{Dom}(P_2(b, V)), \|f\|_{L^2} = 1\},$$

denote the numerical range of the operator $P_2(b, V)$. The inclusion $\sigma(P_2(b, V)) \subseteq \overline{N(P_2(b, V))}$ is well known (see for instance [10, Lemma 9.3.14]).

Proposition 4.1. *There exists $r_0 > 0$ such that for any $0 < r < |\mu| < r_0$, we have:*

$$(4.2) \quad \begin{aligned} & (i) \ \mu \text{ is a discrete eigenvalue of } P_2(b, V) \text{ near zero if and only if } \mu \text{ is a zero of} \\ & \mathcal{D}_\perp(\mu, r) := \det(I + \mathcal{K}_\perp(\mu, r)), \end{aligned}$$

with $\mathcal{K}_\perp(\mu, r)$ a finite-rank operator analytic with respect to μ and satisfying

$$\text{rank } \mathcal{K}_\perp(\mu, r) = \mathcal{O}\left(\text{Tr } \mathbf{1}_{(r, \infty)}(p|V|_{11}p) + 1\right), \quad \|\mathcal{K}_\perp(\mu, r)\| = \mathcal{O}(r^{-1}),$$

where the \mathcal{O} 's are uniform with respect to r, μ .

$$(4.3) \quad \begin{aligned} & (ii) \text{ Furthermore, if } \mu \text{ is a discrete eigenvalue of } P_2(b, V) \text{ near zero, then,} \\ & \text{mult}(\mu) = \text{Ind}_{\mathcal{C}}(I + \mathcal{K}_\perp(\cdot, r)) = m(\mu), \\ & \mathcal{C} \text{ being chosen as in (3.16), and } m(\mu) \text{ being the multiplicity of } \mu \text{ as a zero of } \mathcal{D}_\perp(\cdot, r). \\ & (iii) \text{ If } \mu \text{ verifies } \text{dist}(\mu, \overline{N(P_2(b, V))}) > \varsigma > 0, \varsigma = \mathcal{O}(1), \text{ then } I + \mathcal{K}_\perp(\mu, r) \text{ is invertible} \\ & \text{and satisfies } \|(I + \mathcal{K}_\perp(\mu, r))^{-1}\| = \mathcal{O}(\varsigma^{-1}), \text{ where the } \mathcal{O} \text{ is uniform with respect to } r, \\ & \mu \text{ and } \varsigma. \end{aligned}$$

Proof. (i)-(ii): By Proposition 3.2, $\mu \mapsto \mathcal{A}_\perp(\mu)$ is holomorphic near zero with values in $\mathcal{S}_q(L^2(\mathbb{R}^2))$. Then, for r_0 small enough, there exists a finite-rank operator $\mathcal{A}_{0,\perp}$ independent of μ , and an operator $\tilde{\mathcal{A}}_\perp(\mu) \in \mathcal{S}_q(L^2(\mathbb{R}^2))$ holomorphic near zero satisfying $\|\tilde{\mathcal{A}}_\perp(\mu)\| < \frac{1}{4}$ for $0 < |\mu| \leq r_0$, and

$$\mathcal{A}_\perp(\mu) = \mathcal{A}_{0,\perp} + \tilde{\mathcal{A}}_\perp(\mu).$$

Set $\mathcal{B}_\perp := |V|^{\frac{1}{2}} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}}$ and write

$$(4.4) \quad \mathcal{B}_\perp = \mathcal{B}_\perp \mathbf{1}_{[0, \frac{1}{2}r]}(\mathcal{B}_\perp) + \mathcal{B}_\perp \mathbf{1}_{(\frac{1}{2}r, \infty)}(\mathcal{B}_\perp).$$

It is easy to check that $\left\| -\frac{\tilde{J}}{\mu} \mathcal{B}_\perp \mathbf{1}_{[0, \frac{1}{2}r]}(\mathcal{B}_\perp) + \tilde{\mathcal{A}}_\perp(\mu) \right\| < \frac{3}{4}$ for $0 < r < |\mu| < r_0$. Consequently, we have

$$(4.5) \quad (I + \mathcal{T}_{V,\perp}(\mu)) = (I + \mathcal{K}_\perp(\mu, r)) \left(I - \frac{\tilde{J}}{\mu} \mathcal{B}_\perp \mathbf{1}_{[0, \frac{1}{2}r]}(\mathcal{B}_\perp) + \tilde{\mathcal{A}}_\perp(\mu) \right),$$

where $\mathcal{K}_\perp(\mu, r)$ is defined by

$$\mathcal{K}_\perp(\mu, r) := \left(-\frac{\tilde{J}}{\mu} \mathcal{B}_\perp \mathbf{1}_{(\frac{1}{2}r, \infty)}(\mathcal{B}_\perp) + \mathcal{A}_{0,\perp} \right) \left(I - \frac{\tilde{J}}{\mu} \mathcal{B}_\perp \mathbf{1}_{[0, \frac{1}{2}r]}(\mathcal{B}_\perp) + \tilde{\mathcal{A}}_\perp(\mu) \right)^{-1}.$$

It is a finite-rank operator of order

$$\mathcal{O}\left(\text{Tr } \mathbf{1}_{(\frac{1}{2}r, \infty)}(\mathcal{B}_\perp) + 1\right) = \mathcal{O}\left(\text{Tr } \mathbf{1}_{(r, \infty)}(p|V|_{11}p) + 1\right),$$

taking into account (3.19). Moreover, its norm is of order $\mathcal{O}(|\mu|^{-1}) = \mathcal{O}(r^{-1})$. Since we have $\| -\frac{\tilde{J}}{\mu} \mathcal{B}_\perp \mathbf{1}_{[0, \frac{1}{2}r]}(\mathcal{B}_\perp) + \tilde{\mathcal{A}}_\perp(\mu) \| < 1$ for $0 < r < |\mu| < r_0$, then [19, Theorem 4.4.3] implies that

$$\text{Ind}_{\mathcal{E}} \left(I - \frac{\tilde{J}}{\mu} \mathcal{B}_\perp \mathbf{1}_{[0, \frac{1}{2}r]}(\mathcal{B}_\perp) + \tilde{\mathcal{A}}_\perp(\mu) \right) = 0.$$

Therefore, Property (10.4) applied to (4.5) together with Proposition 3.1 give (4.3). Furthermore, it follows that μ is a discrete eigenvalue of $P_2(b, V)$ near zero if and only if μ is a zero of $\mathcal{D}_\perp(\cdot, r)$.

(iii): Equality (4.5) implies that we have

$$(4.6) \quad I + \mathcal{K}_\perp(\mu, r) = (I + \mathcal{T}_{V, \perp}(\mu)) \left(I - \frac{\tilde{J}}{\mu} \mathcal{B}_\perp \mathbf{1}_{[0, \frac{1}{2}r]}(\mathcal{B}_\perp) + \tilde{\mathcal{A}}_\perp(\mu) \right)^{-1},$$

for $0 < r < |\mu| < r_0$. From the resolvent equation, it is easy to deduce that

$$\left(I + \tilde{J}|V|^{1/2}(P_2(b, 0) - \mu)^{-1}|V|^{1/2} \right) \left(I - \tilde{J}|V|^{1/2}(P_2(b, V) - \mu)^{-1}|V|^{1/2} \right) = I.$$

Then, if μ belongs to resolvent set of $P_2(b, V)$, we have

$$(I + \mathcal{T}_{V, \perp}(\mu))^{-1} = I - \tilde{J}|V|^{1/2}(P_2(b, V) - \mu)^{-1}|V|^{1/2}.$$

This together with (4.6) imply that $I + \mathcal{K}_\perp(\mu, r)$ is invertible for $0 < r < |\mu| < r_0$. So, using [10, Lemma 9.3.14], we obtain

$$\begin{aligned} \left\| (I + \mathcal{K}_\perp(\mu, r))^{-1} \right\| &= \mathcal{O} \left(1 + \left\| |V|^{1/2}(P_2(b, V) - \mu)^{-1}|V|^{1/2} \right\| \right) \\ &= \mathcal{O} \left(1 + \text{dist}(\mu, \overline{N(P_2(b, V))})^{-1} \right) = \mathcal{O} \left(\varsigma^{-1} \right), \end{aligned}$$

whenever $\text{dist}(\mu, \overline{N(P_2(b, V))}) > \varsigma > 0$, $\varsigma = \mathcal{O}(1)$, and the proof is complete. \square

4.2. Back to the proof of Theorem 2.1. Proposition 4.1 implies that for $0 < r < |\mu| < r_0$, we have

$$\begin{aligned} (4.7) \quad \mathcal{D}_\perp(\mu, r) &= \prod_{j=1}^{\mathcal{O}(\text{Tr} \mathbf{1}_{(r, \infty)}(p|V|_{11}p) + 1)} (1 + \lambda_j(\mu, r)) \\ &= \mathcal{O}(1) \exp \left(\mathcal{O}(\text{Tr} \mathbf{1}_{(r, \infty)}(p|V|_{11}p) + 1) |\ln r| \right), \end{aligned}$$

$\lambda_j(\mu, r)$ being the eigenvalues of $\mathcal{K}_\perp := \mathcal{K}_\perp(\mu, r)$ satisfying $|\lambda_j(\mu, r)| = \mathcal{O}(r^{-1})$. Let $\mu \in D(0, \epsilon)^*$ satisfy $\text{dist}(\mu, \overline{N(P_2(b, V))}) > \varsigma > 0$ and $0 < r < |\mu| < r_0$. Then, we have

$$\mathcal{D}_\perp(\mu, r)^{-1} = \det(I + \mathcal{K}_\perp)^{-1} = \det(I - \mathcal{K}_\perp(I + \mathcal{K}_\perp)^{-1}).$$

Similarly to (4.7), it can be shown that

$$(4.8) \quad |\mathcal{D}_\perp(\mu, r)| \geq C \exp \left(-C(\text{Tr} \mathbf{1}_{(r, \infty)}(p|V|_{11}p) + 1)(|\ln \varsigma| + |\ln r|) \right),$$

so that for $\frac{1}{4}r < \varsigma < 2r$, $0 < r \ll 1$, we obtain

$$(4.9) \quad -\ln |\mathcal{D}_\perp(\mu, r)| \leq C \text{Tr} \mathbf{1}_{(r, \infty)}(p|V|_{11}p) |\ln r| + \mathcal{O}(1).$$

Consider the discrete eigenvalues $\mu \in \{r < |\mu| < 2r\} \subset D(0, \zeta)^*$ with $r > 0$ such that $r < \|V\| < \frac{3}{2}r$. Thanks to their discontinuous distribution, there exists a simply connected sub-domain Δ of $\{r < |\mu| < 2r\}$ containing all the eigenvalues μ and such that $\sigma_{\text{disc}}(P_2(b, V)) \cap \partial\Delta = \emptyset$. Note that in view of the definition (4.1) of the numerical range $N(P_2(b, V))$ of the operator $P_2(b, V)$, we have

$$(4.10) \quad N(P_2(b, V)) \subseteq \{\mu \in \mathbb{C} : |\Im(\mu)| \leq \|V\|\}.$$

Then, Theorem 2.1 holds by applying the Jensen Lemma 11.1 with the function $g(\mu) := \mathcal{D}_\perp(r\mu, r)$, $\mu \in \Delta/r$, with some $\mu_0 \in \Delta/r$ satisfying $\text{dist}(r\mu_0, \overline{N(P_2(b, V))}) \geq \varsigma > \frac{1}{4}r$, $\varsigma < 2r$, and by using (4.7) and (4.9).

5. PROOF OF THEOREM 2.2: LOCALISATION AND ASYMPTOTICS EXPANSIONS

First, we have to rephrase Proposition 3.1 with respect to the characteristic value terminology (see Definition 11.1).

Proposition 5.1. *For $\mu \in D(0, \epsilon)^*$, the following assertions are equivalent:*

- (i) μ is a discrete eigenvalue of $P_2(b, V)$,
- (ii) μ is a characteristic value of $I + \mathcal{T}_{V, \perp}(\cdot)$.

Moreover, the multiplicity of μ as a discrete eigenvalue coincides with its multiplicity as a characteristic value defined by (11.2).

Proof of assertion (i) of Theorem 2.2: From Proposition 5.1 and according to (ii) of Remark 3.1, we reduce the investigation of the discrete eigenvalues $\mu \in D(0, \epsilon)^*$ to that of the characteristic values of

$$I + \mathcal{T}_{V, \perp}(\mu) = I - \eta \frac{\mathcal{A}_\perp(\mu)}{\mu},$$

the operator $\mathcal{A}_\perp(\mu)$ being defined by (2.8). In particular, $\pm \mathcal{A}_\perp(0) = |W|^{\frac{1}{2}} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} |W|^{\frac{1}{2}}$ for $J = \pm$. Thus, assertion (i) of Theorem 2.1 holds by (i) and (ii) of Lemma 11.2 with $z = \pm\mu/\eta$. To be more precise, near zero, the discrete eigenvalues μ verify for any $\delta > 0$

$$(5.1) \quad \pm \Re\left(\frac{\mu}{\eta}\right) \geq 0, \quad \mu \in \pm \eta \overline{\Gamma^\delta(r, r_0)},$$

the sector $\Gamma^\delta(r, r_0)$ being defined by (2.4).

Proof of assertion (ii) of Theorem 2.2: The above proof of (i) of Theorem 2.2 together with Proposition 5.1, imply that the discrete eigenvalues μ near zero are the characteristic values $\mu \in \mathcal{Z}(D(0, \epsilon)^*)$ of $I + \mathcal{T}_{V, \perp}(\cdot)$ concentrated in the sectors $\{\mu \in D(0, \epsilon)^* : \pm\mu/\eta \in \Gamma^\delta(r, r_0)\}$, for any $\delta > 0$. In particular, we obtain

$$(5.2) \quad \sum_{\substack{\mu \in \sigma_{\text{disc}}(P_2(b, V)) \\ r < |\mu| < r_0}} \text{mult}(\mu) = \sum_{\substack{\mu \in \mathcal{Z}(D(0, \epsilon)^*) \\ \pm\mu/\eta \in \Gamma^\delta(r, r_0)}} \text{mult}(\mu) + \mathcal{O}(1) = \mathcal{N}(\Gamma^\delta(r, r_0)) + \mathcal{O}(1),$$

the quantity $\mathcal{N}(\bullet)$ being defined by (11.3). If $n(\bullet)$ is the quantity defined by (11.4) with $T(0) = \pm \mathcal{A}_\perp(0)$, then by using (3.19), we get

$$(5.3) \quad n([r, r_0]) = \text{Tr} \mathbf{1}_{(r, \infty)} \left(|W|^{\frac{1}{2}} \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} |W|^{\frac{1}{2}} \right) + \mathcal{O}(1) = \text{Tr} \mathbf{1}_{(r, \infty)} (p|W|_{11}p) + \mathcal{O}(1).$$

Thus, (ii) of Theorem 2.2 follows from (iii) of Lemma 11.2 together with (5.2) and (5.3).

Proof of assertion (iii) of Theorem 2.2: If we have $\Phi(r) = r^{-\gamma}$, or $\Phi(r) = |\ln r|^\gamma$, or $\Phi(r) = (\ln |\ln r|)^{-1} |\ln r|$ for some $\gamma > 0$, then it can be checked that

$$\phi(r(1 \pm \nu)) = \phi(r)(1 + o(1) + \mathcal{O}(\nu)),$$

for any $\nu > 0$ small enough. Then, (iii) of Theorem 2.2 holds by (iv) of Lemma 11.2 combined with (5.2) and (5.3). This completes the proof.

6. DISCRETE EIGENVALUES FOR THE 3D PROBLEM

6.1. Preliminary results. Define $P := p \otimes 1$, $Q := I - P$, and introduce the orthogonal projections in $L^2(\mathbb{R}^3)$

$$(6.1) \quad P := \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}, \quad Q := I - P = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}.$$

For $z \notin \sigma(P_3(b, 0))$, on account of (1.5) with $n = 3$ and Proposition 1.1, we have

$$(6.2) \quad (P_3(b, 0) - z)^{-1}P = \begin{pmatrix} p \otimes \mathcal{R}(z) & 0 \\ 0 & 0 \end{pmatrix},$$

with the resolvent $\mathcal{R}(z) := \left(-\frac{d^2}{dX_\parallel^2} - z \right)^{-1}$ acting in $L^2(\mathbb{R})$. Then, for any $z \in \mathbb{C} \setminus [0, +\infty)$, we have

$$(6.3) \quad (P_3(b, 0) - z)^{-1} = (p \otimes \mathcal{R}(z)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (P_3(b, 0) - z)^{-1}Q.$$

We have the following lemma:

Lemma 6.1. *For given $U \in L^q(\mathbb{R}^2)$ and $G \in L^q(\mathbb{R})$, $q \in [2, +\infty)$, the operator-valued function*

$$\mathbb{C} \setminus [0, +\infty) \ni z \mapsto UG(P_3(b, 0) - z)^{-1}P$$

is holomorphic with values in $\mathcal{S}_q(L^2(\mathbb{R}^3))$. Moreover,

$$(6.4) \quad \left\| UG(P_3(b, 0) - z)^{-1}P \right\|_{\mathcal{S}_q}^q \leq C \frac{b_0 e^{2\text{osc } \tilde{\varphi}}}{2\pi} \|U\|_{L^q}^q \|G\|_{L^q}^q \left(1 + \frac{|z + 1|}{\text{dist}(z, [0, +\infty))} \right)^q,$$

where $C = C(q)$ is a constant depending only on q .

Proof. The holomorphicity on $\mathbb{C} \setminus [0, +\infty)$ is trivial. Let us prove (6.4).

Thanks to (6.2), we have

$$(6.5) \quad UG(P_3(b, 0) - z)^{-1}P = (Up \otimes G\mathcal{R}(z)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As in (3.5), we have

$$(6.6) \quad \|Up\|_{\mathcal{S}_q}^q \leq \frac{b_0 e^{2\text{osc } \tilde{\varphi}}}{2\pi} \|U\|_{L^q}^q.$$

From the estimate

$$(6.7) \quad \|G\mathcal{R}(z)\|_{\mathcal{S}_q}^q \leq \left\| G \left(-\frac{d^2}{dX_{\parallel}^2} + 1 \right)^{-1} \right\|_{\mathcal{S}_q}^q \left\| \left(-\frac{d^2}{dX_{\parallel}^2} + 1 \right) \mathcal{R}(z) \right\|_{\mathcal{S}_q}^q,$$

we obtain by the Spectral mapping theorem

$$(6.8) \quad \left\| \left(-\frac{d^2}{dX_{\parallel}^2} + 1 \right) \mathcal{R}(z) \right\|_{\mathcal{S}_q}^q \leq \sup_{s \in [0, +\infty)}^q \left| \frac{s+1}{s-z} \right| \leq \left(1 + \frac{|z+1|}{\text{dist}(z, [0, +\infty))} \right)^q,$$

and by the standard criterion [37, Theorem 4.1], we obtain

$$(6.9) \quad \left\| G \left(-\frac{d^2}{dX_{\parallel}^2} + 1 \right) \right\|_{\mathcal{S}_q}^q \leq C \|G\|_{L^q}^q \left\| (|\cdot|^2 + 1)^{-1} \right\|_{L^q}^q.$$

Then, (6.4) follows by putting together (6.5), (6.6), (6.7), (6.8) and (6.9). \square

The next lemma is just the analogue of Lemma 3.2 in dimension three. It can be proved in a similar way by taking into account the appropriate modifications. For this reason, to simplify our exposition, its proof will be omitted.

Lemma 6.2. *For a given $g \in L^q(\mathbb{R}^3)$, $q \in [2, +\infty)$, the operator-valued function*

$$\mathbb{C} \setminus [\zeta, +\infty) \ni z \longmapsto g(P_3(b, 0) - z)^{-1} \mathbf{Q}$$

is holomorphic with values in $\mathcal{S}_q(L^2(\mathbb{R}^3))$. Moreover,

$$(6.10) \quad \|g(P_3(b, 0) - z)^{-1} \mathbf{Q}\|_{\mathcal{S}_q}^q \leq C \|g\|_{L^q}^q \left(1 + \frac{|z+1|}{\text{dist}(z, [\zeta, +\infty))} \right)^q,$$

where $C = C(q)$ is a constant depending only on q .

Throughout this article, we will use the following choice of the complex square root

$$(6.11) \quad \mathbb{C} \setminus (-\infty, 0] \xrightarrow{\sqrt{\cdot}} \mathbb{C}_+.$$

For $0 < \kappa < \sqrt{\zeta}$, let $D_{\pm}(0, \kappa^2)$ be the half-rings defined by (2.17). Put the change of variables $z = k^2$ and define the domains

$$(6.12) \quad \mathcal{D}_{\pm}^*(\kappa) := \{k \in \mathbb{C}_{\pm} : 0 < |k| < \kappa : \Re(k) > 0\}.$$

Under the above considerations, $D_{\pm}(0, \kappa^2)$ can be parametrized by $z = z(k) := k^2$, with $k \in \mathcal{D}_{\pm}^*(\kappa)$ respectively (see Figure 6.1 below):

Now, Assumption (C1) given by (1.11) implies that there exists a bounded operator \mathcal{V} such that $|V|^{\frac{1}{2}} = \mathcal{V} G_{\perp}^{\frac{1}{2}} G_{\parallel}^{\frac{1}{2}}$, with $G_{\perp}^{\frac{1}{2}}$ and $G_{\parallel}^{\frac{1}{2}}$ satisfying (1.11). Then, the boundedness of V , identities (6.2), (6.3), together with Lemmas 6.1 and 6.2 give the following:

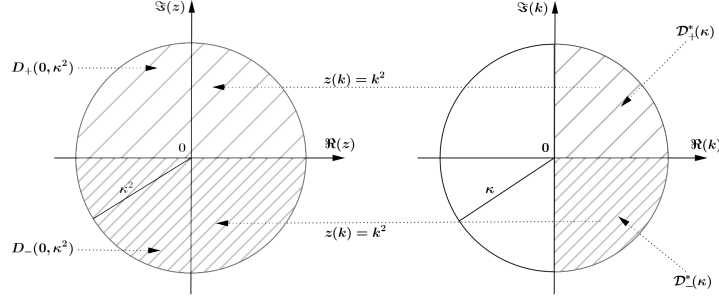


FIGURE 6.1. Images $D_{\pm}(0, \kappa^2)$ of $\mathcal{D}_{\pm}^*(\kappa)$ by the local parametrisation $z(k) = k^2$.

Lemma 6.3. *The operator-valued functions*

$$\mathcal{D}_{\pm}^*(\kappa) \ni k \mapsto \mathcal{T}_V(z(k)) := \tilde{J}|V|^{\frac{1}{2}}(P_3(b, 0) - z(k))^{-1}|V|^{\frac{1}{2}}, \quad z(k) := k^2,$$

are holomorphic with values in $\mathcal{S}_q(L^2(\mathbb{R}^3))$, where \tilde{J} is defined by the polar decomposition $V = \tilde{J}|V|$ of V .

6.2. Reduction of the problem. Similarly to Lemma 6.3, we can show that $V(P_3(b, 0) - z(\cdot))^{-1}$ is holomorphic in $\mathcal{D}_{\pm}^*(\kappa)$ with values in $\mathcal{S}_q(L^2(\mathbb{R}^3))$. Then, as in (3.15), we have

$$(6.13) \quad z(k) \in \sigma_{\text{disc}}(P_3(b, V)) \Leftrightarrow \det_{[q]} \left(I + V(P_3(b, 0) - z(k))^{-1} \right) = 0.$$

We are thus led to the following proposition:

Proposition 6.1. *Let $\mathcal{T}_V(z(k))$ be the operator defined in Lemma 6.3. Then, the following assertions are equivalent:*

- (i) $z(k) := k^2 \in D_{\pm}(0, \kappa^2)$ is a discrete eigenvalue of $P_3(b, V)$,
- (ii) $\det_{[q]} (I + \mathcal{T}_V(z(k))) = 0$,
- (iii) -1 is an eigenvalue of $\mathcal{T}_V(z(k))$.

Moreover,

$$(6.14) \quad \text{mult}(z(k)) = \text{Ind}_{\mathcal{C}} \left(I + \mathcal{T}_V(z(\cdot)) \right),$$

where \mathcal{C} is a small contour positively oriented containing k as the unique point $k \in \mathcal{D}_{\pm}^*(\kappa)$ verifying $z(k) \in D_{\pm}(0, \kappa^2)$ is a discrete eigenvalue of $P_3(b, V)$.

Proof. The proof is similar to that of Proposition 3.1, taking into account the appropriate modifications. \square

6.3. Decomposition of the weighted resolvent. Our goal in this section is to decompose $\mathcal{T}_V(z(k))$ (defined just above), as a sum of a singular part at $k = 0$, and a holomorphic part in $\mathcal{D}_{\pm}^*(\kappa)$ continuous near $k = 0$ with values in $\mathcal{S}_q(L^2(\mathbb{R}^3))$. The potential V is supposed to verify Assumption (C1).

Observe that according to the choice (6.11) of the complex square root, we respectively have $\sqrt{k^2} = \pm k$ for $k \in \mathcal{D}_\pm^*(\kappa)$. By identity (6.3), we have

$$(6.15) \quad \mathcal{T}_V(z(k)) = \tilde{J}|V|^{\frac{1}{2}}p \otimes \mathcal{R}(z(k)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}} + \tilde{J}|V|^{\frac{1}{2}}(P_3(b, 0) - z(k))^{-1}Q|V|^{\frac{1}{2}}.$$

Let us focus on the first term of the r.h.s. of (6.15) and define respectively G_\pm as the multiplication operators by the functions $\mathbb{R} \ni X_\parallel \mapsto G^{\pm \frac{1}{2}}(X_\parallel)$. Then, we get

$$(6.16) \quad \tilde{J}|V|^{\frac{1}{2}}p \otimes \mathcal{R}(z(k)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}} = \tilde{J}|V|^{\frac{1}{2}}G_-p \otimes G_+\mathcal{R}(z(k))G_+ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G_-|V|^{\frac{1}{2}}.$$

For $z \in \mathbb{C} \setminus [0, +\infty)$, $\mathcal{R}(z) := (-\frac{d^2}{dX_\parallel^2} - z)^{-1}$ admits the integral kernel

$$(6.17) \quad I_z(X_\parallel, X'_\parallel) := -\frac{e^{i\sqrt{z}|X_\parallel - X'_\parallel|}}{2i\sqrt{z}}.$$

Then, the integral kernel of the operator $G_+\mathcal{R}(z(k))G_+$ is given by

$$(6.18) \quad \pm G^{\frac{1}{2}}(X_\parallel) \frac{ie^{\pm ik|X_\parallel - X'_\parallel|}}{2k} G^{\frac{1}{2}}(X'_\parallel), \quad k \in \mathcal{D}_\pm^*(\kappa).$$

With the help of (6.18), we can write

$$(6.19) \quad G_+\mathcal{R}(z(k))G_+ = \pm \frac{1}{k}a + b(k), \quad k \in \mathcal{D}_\pm^*(\kappa),$$

$a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ being the rank-one operator given by

$$(6.20) \quad a(u) := \frac{i}{2}\langle u, G^{\frac{1}{2}}(\cdot) \rangle G^{\frac{1}{2}}(X_\parallel),$$

and $b(k)$ being the operator with integral kernel given by

$$(6.21) \quad \pm G^{\frac{1}{2}}(X_\parallel) i \frac{e^{\pm ik|X_\parallel - X'_\parallel|} - 1}{2k} G^{\frac{1}{2}}(X'_\parallel).$$

Moreover, it is easy to observe that we have $-2ia = c^*c$ with $c : L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by $c(u) := \langle u, G^{\frac{1}{2}}(\cdot) \rangle$, so that $c^* : \mathbb{C} \longrightarrow L^2(\mathbb{R})$ is given by $c^*(\lambda) = \lambda G^{\frac{1}{2}}(\cdot)$. Putting this together with (6.19) and (6.21), we get

$$(6.22) \quad p \otimes G_+\mathcal{R}(z(k))G_+ = \pm \frac{i}{2k}p \otimes c^*c + p \otimes s(k), \quad k \in \mathcal{D}_\pm^*(\kappa),$$

where $s(k)$ is the operator acting from $G^{\frac{1}{2}}(X_\parallel)L^2(\mathbb{R})$ to $G^{-\frac{1}{2}}(X_\parallel)L^2(\mathbb{R})$ with integral kernel

$$(6.23) \quad \pm \frac{1 - e^{\pm ik|X_\parallel - X'_\parallel|}}{2ik}.$$

Equality (6.16) combined with (6.22) give for $k \in \mathcal{D}_\pm^*(\kappa)$

$$(6.24) \quad \begin{aligned} & \tilde{J}|V|^{\frac{1}{2}}p \otimes \mathcal{R}(z(k)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}} \\ &= \pm \frac{i\tilde{J}}{2k}|V|^{\frac{1}{2}}G_-(p \otimes c^*c) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G_-|V|^{\frac{1}{2}} + \tilde{J}|V|^{\frac{1}{2}}G_-p \otimes s(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G_-|V|^{\frac{1}{2}}. \end{aligned}$$

Finally, we get for $k \in \mathcal{D}_\pm^*(\kappa)$

$$(6.25) \quad \tilde{J}|V|^{\frac{1}{2}}p \otimes \mathcal{R}(z(k)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}} = \pm \frac{i\tilde{J}}{k} K^* K + \tilde{J}|V|^{\frac{1}{2}} G_- p \otimes s(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G_- |V|^{\frac{1}{2}},$$

K being defined by

$$(6.26) \quad K := \frac{1}{\sqrt{2}}(p \otimes c) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G_- |V|^{\frac{1}{2}}.$$

More precisely, recalling that $X_\perp := (x, y) \in \mathbb{R}^2$, we have $K : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^2)$ with

$$(K\psi)(X_\perp) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} dX'_\perp dX'_\parallel \mathcal{P}_b(X_\perp, X'_\perp) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}}(X'_\perp, X'_\parallel) \psi(X'_\perp, X'_\parallel),$$

where $\mathcal{P}_b(\cdot, \cdot)$ is the integral kernel of the orthogonal projection $p := p(b)$ (see [21, Theorem 2.3]). Obviously, $K^* : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^3)$ is given by

$$(K^*\varphi)(X_\perp, X_\parallel) = \frac{1}{\sqrt{2}} |V|^{\frac{1}{2}}(X_\perp, X_\parallel) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (p\varphi)(X_\perp).$$

Therefore, it can be checked that the operator $KK^* : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$ satisfies

$$(6.27) \quad KK^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} p \mathbf{V}_{11} p,$$

where \mathbf{V}_{11} is the multiplication operator by the function (also noted) \mathbf{V}_{11} defined by (2.14).

For $\lambda \in \mathbb{R}_+ \setminus \{0\}$, we define $(-\frac{d^2}{dX_\parallel^2} - \lambda)^{-1}$ as the operator with integral kernel

$$(6.28) \quad I_\lambda(X_\parallel, X'_\parallel) := \lim_{\delta \downarrow 0} I_{\lambda+i\delta}(X_\parallel, X'_\parallel) = \frac{ie^{i\sqrt{\lambda}|X_\parallel - X'_\parallel|}}{2\sqrt{\lambda}},$$

$I_z(\cdot)$ being defined by (6.17). Then, as in [29, (Proof of) Proposition 4.2], we can show by a limiting absorption principle that the operator-valued function $\overline{\mathcal{D}_\pm^*(\kappa)} \ni k \mapsto G_+ s(k) G_+ \in \mathcal{S}_2(L^2(\mathbb{R}))$ is well defined and continuous. We thus have proved the following:

Proposition 6.2. *For $k \in \mathcal{D}_\pm^*(\kappa)$, we have*

$$(6.29) \quad \mathcal{T}_V(z(k)) = \pm \frac{i\tilde{J}}{k} \mathcal{B} + \mathcal{A}(k), \quad \mathcal{B} := K^* K,$$

where the operator $\mathcal{A}(k) \in \mathcal{S}_q(L^2(\mathbb{R}^3))$ given by

$$\mathcal{A}(k) := \tilde{J}|V|^{\frac{1}{2}} G_- p \otimes s(k) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} G_- |V|^{\frac{1}{2}} + \tilde{J}|V|^{\frac{1}{2}} (P_3(b, 0) - z(k))^{-1} Q |V|^{\frac{1}{2}},$$

is holomorphic in $\mathcal{D}_\pm^*(\kappa)$ and continuous on $\overline{\mathcal{D}_\pm^*(\kappa)}$, with $s(k)$ defined by (6.22).

Remark 6.1.

(i) For any $r > 0$, according to (6.27), we have

$$(6.30) \quad \text{Tr} \mathbf{1}_{(r, \infty)} (K^* K) = \text{Tr} \mathbf{1}_{(r, \infty)} (KK^*) = \text{Tr} \mathbf{1}_{(r, \infty)} (p \mathbf{V}_{11} p).$$

(ii) If V satisfies Assumption (C2) given by (2.20), then Proposition 3.2 holds with \tilde{J} replaced by $J\eta$, $J := \text{sign}(W)$, and \mathbf{V}_{11} replaced by \mathbf{W}_{11} in (6.30).

7. PROOF OF THEOREM 2.3: UPPER BOUNDS

The proof is similar to that of Theorem 2.1.

7.1. A preliminary result. Introduce the numerical range

$$N(P_3(b, V)) := \{ \langle P_3(b, V)f, f \rangle : f \in \text{Dom}(P_3(b, V)), \|f\|_{L^2} = 1 \},$$

satisfying $\sigma(P_3(b, V)) \subseteq \overline{N(P_3(b, V))}$.

Proposition 7.1. *There exists $r_0 > 0$ such that for any $k \in \{0 < r < |k| < r_0\} \cap \mathcal{D}_\pm^*(\kappa)$, we have:*

$$(7.1) \quad (i) \quad z(k) := k^2 \text{ is a discrete eigenvalue of } P_3(b, V) \text{ near zero if and only if } k \text{ is a zero of } \mathcal{D}(k, r) := \det(I + \mathcal{K}(k, r)),$$

with $\mathcal{K}(k, r)$ a finite-rank operator analytic with respect to k and satisfying

$$\text{rank } \mathcal{K}(k, r) = \mathcal{O}(\text{Tr } \mathbf{1}_{(r, \infty)}(p\mathbf{V}_{11}p) + 1), \quad \|\mathcal{K}(k, r)\| = \mathcal{O}(r^{-1}),$$

where the \mathcal{O} 's are uniform with respect to r, k .

$$(7.2) \quad (ii) \quad \text{Furthermore, if } z(k) := k^2 \text{ is a discrete eigenvalue of } P_3(b, V) \text{ near zero, then we have } \text{mult}(z(k)) = \text{Ind}_{\mathcal{C}}(I + \mathcal{K}(\cdot, r)) = m(k),$$

\mathcal{C} being chosen as in (6.14), and $m(k)$ being the multiplicity of k as a zero of $\mathcal{D}(\cdot, r)$.

$$(iii) \quad \text{If } z(k) \text{ satisfies } \text{dist}(z(k), \overline{N(P_3(b, V))}) > \varsigma > 0, \varsigma = \mathcal{O}(1), \text{ then } I + \mathcal{K}(k, r) \text{ is invertible and satisfies } \|(I + \mathcal{K}(k, r))^{-1}\| = \mathcal{O}(\varsigma^{-1}), \text{ where the } \mathcal{O} \text{ is uniform with respect to } r, k \text{ and } \varsigma.$$

Proof. The proof follows by arguing similarly to that of Proposition 4.1, taking into account the appropriate modifications. \square

7.2. Back tot the proof Theorem 2.3. Proposition 7.1 above implies that

$$(7.3) \quad \begin{aligned} \mathcal{D}(k, r) &= \prod_{j=1}^{\mathcal{O}(\text{Tr } \mathbf{1}_{(r, \infty)}(p\mathbf{V}_{11}p) + 1)} (1 + \lambda_j(k, r)) \\ &= \mathcal{O}(1) \exp\left(\mathcal{O}(\text{Tr } \mathbf{1}_{(r, \infty)}(p\mathbf{V}_{11}p) + 1) |\ln r|\right), \end{aligned}$$

for $0 < r < |k| < r_0$, the $\lambda_j(k, r)$ being the eigenvalues of $\mathcal{K} := \mathcal{K}(k, r)$ satisfying $|\lambda_j(k, r)| = \mathcal{O}(r^{-1})$. If $\text{dist}(z(k), \overline{N(P_3(b, V))}) > \varsigma > 0$ with $0 < r < |k| < r_0$, then,

$$\mathcal{D}(k, r)^{-1} = \det(I + \mathcal{K})^{-1} = \det(I - \mathcal{K}(I + \mathcal{K})^{-1}).$$

Similarly to (7.3), we can show that

$$(7.4) \quad |\mathcal{D}(k, r)| \geq C \exp\left(-C(\text{Tr } \mathbf{1}_{(r, \infty)}(p\mathbf{V}_{11}p) + 1)(|\ln \varsigma| + |\ln r|)\right),$$

so that for $r^2 < \varsigma < 4r^2$, $0 < r \ll 1$, we obtain

$$(7.5) \quad -\ln |\mathcal{D}(k, r)| \leq C \text{Tr } \mathbf{1}_{(r, \infty)}(p|V|_{11}p) |\ln r| + \mathcal{O}(1).$$

Consider the domains $\Delta_{\pm} := \left\{ r < |k| < 2r : |\Re(k)| > \sqrt{\frac{\nu}{2}} : |\Im(k)| > \sqrt{\frac{\nu}{2}} \right\} \cap \mathcal{D}_{\pm}^*(\kappa)$ with $0 < r < \sqrt{\|V\|} < \sqrt{\frac{5}{2}}r$ and $0 < \nu < 2r^2$. Since the numerical range of the operator $P_3(b, V)$ satisfies

$$(7.6) \quad N(P_3(b, V)) \subseteq \{z \in \mathbb{C} : |\Im(z)| \leq \|V\|\},$$

then there exists some $k_0 \in \Delta_{\pm}/r$ satisfying $\text{dist}(z(rk_0), \overline{N(P_3(b, V))}) \geq \varsigma > r^2$, $\varsigma < 4r^2$. Then, Theorem 2.3 follows by applying the Jensen Lemma 11.1 with the function $g(k) := \mathcal{D}(rk, r)$, $k \in \Delta_{\pm}/r$, together with (7.3) and (7.5).

8. PROOF OF THEOREM 2.4: SECTORS FREE OF COMPLEX EIGENVALUES AND LOWER BOUNDS

To simplify, we give the proof only for the case $\alpha \in (0, \pi)$. The case $\alpha \in -(0, \pi)$ follows in a similar way by replacing k by $-k$.

(i): For any $\theta > 0$ small enough, set $\delta = \tan(\theta)$ and introduce the sector

$$(8.1) \quad \mathcal{C}_{\delta} := \{k \in \mathbb{C} : -\delta \Im(k) \leq |\Re(k)|\}.$$

According to (ii) of Remark 6.1, for any $\varepsilon > 0$, we have

$$(8.2) \quad I + \mathcal{T}_{\varepsilon V}(z(k)) = I + \frac{i\varepsilon\eta}{k}\mathcal{B} + \varepsilon\mathcal{A}(k), \quad k \in \mathcal{D}_{+}^*(\kappa),$$

\mathcal{B} being a self-adjoint positive operator independent of k , while $\mathcal{A}(k) \in \mathcal{S}_q(L^2(\mathbb{R}^3))$ is holomorphic in $\mathcal{D}_{+}^*(\kappa)$ and continuous on $\overline{\mathcal{D}_{+}^*(\kappa)}$. Since we have

$$I + \frac{i\varepsilon\eta}{k}\mathcal{B} = \frac{i\eta}{k}(\varepsilon\mathcal{B} - ik\eta^{-1}),$$

then it is easy to see that the operator $I + \frac{i\varepsilon\eta}{k}\mathcal{B}$ is invertible for $ik\eta^{-1} \notin \sigma(\varepsilon\mathcal{B})$. Otherwise, it can be shown that we have

$$(8.3) \quad \left\| \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right)^{-1} \right\| \leq \frac{|k\eta^{-1}|}{\sqrt{(\Im(k\eta^{-1}))_{+}^2 + |\Re(k\eta^{-1})|^2}}, \quad r_{+} := \max(r, 0).$$

Therefore, for $k \in \eta\mathcal{C}_{\delta}$, it can be checked that

$$(8.4) \quad \left\| \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right)^{-1} \right\| \leq \sqrt{1 + \delta^{-2}},$$

$k \in \mathcal{D}_{+}^*(\kappa)$. Then, we have

$$(8.5) \quad I + \mathcal{T}_{\varepsilon V}(z(k)) = (I + A(k)) \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right),$$

where

$$(8.6) \quad A(k) := \varepsilon\mathcal{A}(k) \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right)^{-1} \in \mathcal{S}_q(L^2(\mathbb{R}^3)).$$

Since $\mathcal{A}(k) \in \mathcal{S}_q(L^2(\mathbb{R}^3))$ is continuous on $\overline{\mathcal{D}_+^*(\kappa)}$, then there exists a uniform constant $C > 0$ such that $\|\mathcal{A}(k)\| \leq \|\mathcal{A}(k)\|_{\mathcal{S}_q} \leq C$. Putting this together with (8.4) and (8.6), it follows immediately that $I + \mathcal{T}_{\varepsilon V}(z(k))$ is invertible for $k \in \eta\mathcal{C}_\delta$, $k \in \mathcal{D}_+^*(\kappa)$, and

$$(8.7) \quad 0 < \varepsilon < C_0 := (C\sqrt{1 + \delta^{-2}})^{-1}.$$

This means that $z(k)$ is not a discrete eigenvalue.

(ii): Let $(\mu_j)_j$ denote the sequence of the decreasing nonzero eigenvalues of $p\mathbf{W}_{11}p$ taking into account their multiplicity. If Assumption (C3) given by (2.21) is fulfilled, then similarly to [3, (Proof of) Lemma 7], it can be shown that there exists a positive constant ν such that

$$(8.8) \quad \#\{j : \mu_j - \mu_{j+1} > \nu\mu_j\} = \infty.$$

Since the nonzero eigenvalues of \mathcal{B} and $p\mathbf{W}_{11}p$ coincide, then there exists a decreasing sequence of positive numbers $(r_\ell)_\ell$, $r_\ell \searrow 0$, such that

$$(8.9) \quad \text{dist}(r_\ell, \sigma(\mathcal{B})) \geq \frac{\nu r_\ell}{2}, \quad \ell \in \mathbb{N}.$$

Moreover, there exists for any $\ell \in \mathbb{N}$ a path $\tilde{\Sigma}_\ell := \partial\Lambda_\ell$, where

$$(8.10) \quad \Lambda_\ell := \{\tilde{k} \in \mathbb{C} : 0 < |\tilde{k}| < r_0 : |\Im(\tilde{k})| \leq \delta \Re(\tilde{k}) : r_{\ell+1} \leq \Re(\tilde{k}) \leq r_\ell\},$$

(see Figure 8.1), enclosing the eigenvalues of \mathcal{B} lying in $[r_{\ell+1}, r_\ell]$.

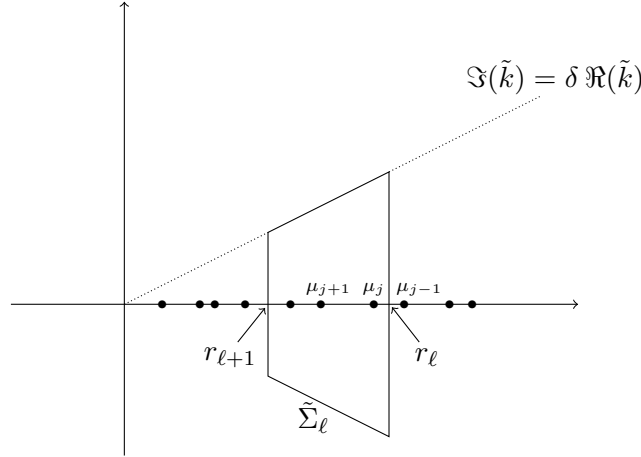


FIGURE 8.1. Representation of the path $\tilde{\Sigma}_\ell = \partial\Lambda_\ell$.

Obviously, the operator $\tilde{k} - \mathcal{B}$ is invertible for $\tilde{k} \in \tilde{\Sigma}_\ell$, and it can be proved that

$$(8.11) \quad \|(\tilde{k} - \mathcal{B})^{-1}\| \leq \frac{1}{\text{dist}(\tilde{k}, \sigma(\mathcal{B}))} = \frac{1}{|\tilde{k}|} \times \frac{|\tilde{k}|}{\text{dist}(\tilde{k}, \sigma(\mathcal{B}))} \leq \frac{C(\delta, \nu)}{|\tilde{k}|},$$

where

$$(8.12) \quad C(\delta, \nu) := \sqrt{1 + \delta^2} \max(\delta^{-1}, (\nu/2)^{-1}).$$

Introduce the path $\Sigma_\ell := -i\varepsilon\eta\tilde{\Sigma}_\ell$. According to the construction of Σ_ℓ and (8.11), we immediately observe that $I + \frac{i\varepsilon\eta}{k}\mathcal{B}$ is invertible for $k \in \Sigma_\ell$ with

$$(8.13) \quad \left\| \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right)^{-1} \right\| \leq C(\delta, \nu).$$

Then, for $k \in \Sigma_\ell$, we have

$$(8.14) \quad I + \frac{i\varepsilon\eta}{k}\mathcal{B} + \varepsilon\mathcal{A}(k) = \left(I + \varepsilon\mathcal{A}(k) \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right)^{-1} \right) \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right).$$

By choosing $0 < \varepsilon \leq \varepsilon_0$ sufficiently small and using Property **e**) of Subsection 9 given by (9.3), we obtain

$$(8.15) \quad \left| \det_{[q]} \left[I + \varepsilon\mathcal{A}(k) \left(I + \frac{i\varepsilon\eta}{k}\mathcal{B} \right)^{-1} \right] - 1 \right| < 1,$$

for any $k \in \Sigma_\ell$. More precisely, if we let C, C_0 be the constants defined by (8.7), $C(\delta, \nu)$ the one defined by (8.12), and Γ_q that defined by (9.3), then (8.15) holds whenever ε satisfies

$$(8.16) \quad 0 < \varepsilon < C^{-1}C(\delta, \nu)^{-1}e^{-\Gamma_q(C_0CC(\delta, \nu)+1)^{[q]}} = C_0C_1(\delta, \nu)^{-1}e^{-\Gamma_q(C_1(\delta, \nu)+1)^{[q]}},$$

where

$$(8.17) \quad C_1(\delta, \nu) := \delta \max \left(\delta^{-1}, (\nu/2)^{-1} \right).$$

Thus, the Rouché Theorem implies that the number of zeros of $\det_{[q]}(I + \frac{i\varepsilon\eta}{k}\mathcal{B} + \varepsilon\mathcal{A}(k))$ enclosed in $\{z(k) \in D_+(0, \kappa^2) : k \in \Lambda_\ell\}$ taking into account their multiplicity, is equal to that of $\det_{[q]}(I + \frac{i\varepsilon\eta}{k}\mathcal{B})$ enclosed in $\{z(k) \in D_+(0, \kappa^2) : k \in \Lambda_\ell\}$ taking into account their multiplicity. This number is equal to $\text{Tr} \mathbf{1}_{[r_{\ell+1}, r_\ell]}(p\mathbf{W}_{11}p)$. Hence, thanks to Proposition 6.1 and Property (10.4) applied to (8.14), bound (2.25) follows immediately since the zeros of $\det_{[q]}(I + \frac{i\varepsilon\eta}{k}\mathcal{B} + \varepsilon\mathcal{A}(k))$ are the discrete eigenvalues of $P_3(b, \varepsilon V)$ taking into account their multiplicity. From the fact that the sequence $(r_\ell)_\ell$ is infinite tending to zero, it follows the infiniteness of the number of the discrete eigenvalues claimed. This concludes the proof of Theorem 2.4.

9. APPENDIX A 1: SCHATTEN-VON NEUMANN IDEALS AND REGULARIZED DETERMINANTS

For the convenience of the reader, we repeat the relevant material from Reed-Simon [30], Simon [36, 37], and Gohberg-Goldberg-Krupnik [20], thus making our exposition self-contained.

Let \mathcal{H} be a separable Hilbert space and $\mathcal{S}_\infty(\mathcal{H})$ be the set of compact linear operators on \mathcal{H} . Denote by $s_k(T)$ the k -th singular value of $T \in \mathcal{S}_\infty(\mathcal{H})$. The Schatten-von Neumann classes are defined by

$$(9.1) \quad \mathcal{S}_q(\mathcal{H}) := \left\{ T \in \mathcal{S}_\infty(\mathcal{H}) : \|T\|_{\mathcal{S}_q}^q := \sum_k s_k(T)^q < +\infty \right\}, \quad q \in [1, +\infty).$$

To simplify, we will write \mathcal{S}_q when no confusion can arise. For $[q] := \min \{n \in \mathbb{N} : n \geq q\}$ and $T \in \mathcal{S}_q$, the regularized determinant is defined by

$$(9.2) \quad \det_{[q]}(I - T) := \prod_{\mu \in \sigma(T)} \left[(1 - \mu) \exp \left(\sum_{k=1}^{[q]-1} \frac{\mu^k}{k} \right) \right].$$

Here are some elementary properties about this determinant (see for instance [36]):

- a) $\det_{[q]}(I) = 1$.
- b) For $A, B \in \mathcal{L}(\mathcal{H})$ the class of bounded linear operators on \mathcal{H} , if AB and BA belong to \mathcal{S}_q , then $\det_{[q]}(I - AB) = \det_{[q]}(I - BA)$.
- c) $I - T$ is invertible if and only if $\det_{[q]}(I - T) \neq 0$.
- d) If $T : D \rightarrow \mathcal{S}_q$ is a holomorphic operator-valued function in a domain D , then so is $\det_{[q]}(I - T(\cdot))$ in D .
- e) $\det_{[q]}(I - T)$ is Lipschitz as function on \mathcal{S}_q uniformly on balls. Explicitly, we have

$$(9.3) \quad |\det_{[q]}(I - T_1) - \det_{[q]}(I - T_2)| \leq \|T_1 - T_2\|_{\mathcal{S}_q} e^{\Gamma_q(\|T_1\|_{\mathcal{S}_q} + \|T_2\|_{\mathcal{S}_q} + 1)^{[q]}},$$

by [36, Theorem 6.5], for some constant $\Gamma_q > 0$.

10. APPENDIX A 2: INDEX OF A FINITE MEROMORPHIC OPERATOR-VALUED FUNCTION

The space \mathcal{H} and the class $\mathcal{L}(\mathcal{H})$ are defined as in Appendix A 1. We have the following definition from [19, Definition 4.1.1].

Definition 10.1. *Let \mathcal{U} be a neighbourhood of a fixed point $w \in \mathbb{C}$, and $F : \mathcal{U} \setminus \{w\} \rightarrow \mathcal{L}(\mathcal{H})$ be a holomorphic operator-valued function. The function F is said to be finite meromorphic at w if its Laurent expansion at w has the form*

$$(10.1) \quad F(z) = \sum_{n=m}^{+\infty} (z - w)^n A_n, \quad m > -\infty,$$

where for $m < 0$, the operators A_m, \dots, A_{-1} are of finite rank. Moreover, if A_0 is a Fredholm operator, then the function F is said to be Fredholm at w . In that case, the Fredholm index of A_0 is called the Fredholm index of F at w .

If a function f is holomorphic in a neighbourhood of a contour \mathcal{C} (positively oriented), its index with respect to this contour is defined by

$$(10.2) \quad \text{ind}_{\mathcal{C}} f := \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{f'(z)}{f(z)} dz.$$

Let us point out that if f is holomorphic in a domain D with $\partial D = \mathcal{C}$, then thanks to the residues theorem, $\text{ind}_{\mathcal{C}} f$ coincides with the number of zeros of f in D taking into account their multiplicity.

In what follows below, $\text{GL}(\mathcal{H})$ denotes the class of invertible linear operators on the Hilbert space \mathcal{H} . Let $D \subseteq \mathbb{C}$ be a connected domain, $Z \subset D$ be a pure point and closed subset, and $A : \overline{D} \setminus Z \rightarrow \text{GL}(\mathcal{H})$ be a finite meromorphic operator-valued function which is Fredholm at each point of Z . The index of A with respect to the contour ∂D is defined by

$$(10.3) \quad \text{Ind}_{\partial D} A := \frac{1}{2i\pi} \text{Tr} \int_{\partial D} A'(z) A(z)^{-1} dz = \frac{1}{2i\pi} \text{Tr} \int_{\partial D} A(z)^{-1} A'(z) dz,$$

where the operator A does not vanish in the integration contour ∂D . The following properties are well known:

$$(10.4) \quad \text{Ind}_{\partial D} A_1 A_2 = \text{Ind}_{\partial D} A_1 + \text{Ind}_{\partial D} A_2;$$

for $K(z)$ a trace class operator-valued function, we have

$$(10.5) \quad \text{Ind}_{\partial D}(I + K) = \text{ind}_{\partial D} \det(I + K).$$

We refer for instance to [19, Chap. 4] for a deeper discussion on the subject.

11. APPENDIX A 3: JENSEN TYPE INEQUALITY AND CHARACTERISTIC VALUES OF OPERATOR-VALUED FUNCTIONS

The following lemma (see for instance [3, Lemma 6] for a proof) contains a version of the well-known Jensen inequality.

Lemma 11.1. *Let Δ be a simply connected sub-domain of \mathbb{C} and let g be holomorphic in Δ with continuous extension to $\bar{\Delta}$. Assume that there exists $\lambda_0 \in \Delta$ such that $g(\lambda_0) \neq 0$ and $g(\lambda) \neq 0$ for $\lambda \in \partial\Delta$ (the boundary of Δ). Let $\lambda_1, \lambda_2, \dots, \lambda_N \in \Delta$ be the zeros of g repeated according to their multiplicity. For any domain $\Delta' \subset\subset \Delta$, there exists $C' > 0$ such that $N(\Delta', g)$, the number of zeros λ_j of g contained in Δ' , satisfies*

$$(11.1) \quad N(\Delta', g) \leq C' \left(\int_{\partial\Delta} \ln|g(\lambda)| d\lambda - \ln|g(\lambda_0)| \right).$$

Consider a domain \mathcal{D} of \mathbb{C} containing 0, and let $T : \mathcal{D} \rightarrow S_\infty(\mathcal{H})$ be a holomorphic operator-valued function, \mathcal{H} being as above.

Definition 11.1. *For a domain $\Omega \subset \mathcal{D} \setminus \{0\}$, a complex number $z \in \Omega$ is a characteristic value of $z \mapsto \mathcal{T}(z) := I - \frac{T(z)}{z}$, if $\mathcal{T}(z)$ is not invertible. The multiplicity of a characteristic value z_0 is defined by*

$$(11.2) \quad \text{mult}(z_0) := \text{Ind}_{\mathcal{C}}(I - \mathcal{T}(\cdot)),$$

where \mathcal{C} is a small contour positively oriented containing z_0 as the unique point z satisfying $\mathcal{T}(z)$ is not invertible.

Define

$$\mathcal{Z}(\Omega) := \{z \in \Omega : \mathcal{T}(z) \text{ is not invertible}\}.$$

Once there exists $z_0 \in \Omega$ satisfying $\mathcal{T}(z_0)$ is not invertible, then by the analytic Fredholm theorem, the set $\mathcal{Z}(\Omega)$ is pure point. Hence, we set

$$(11.3) \quad \mathcal{N}(\Omega) := \#\mathcal{Z}(\Omega).$$

In the sequel, we suppose that the operator $T(0)$ is self-adjoint and we put

$$(11.4) \quad n(\omega) := \text{Tr } \mathbf{1}_\omega(T(0)),$$

the number of eigenvalues of $T(0)$ lying in the interval $\omega \subset \mathbb{R}^*$, taking into account their multiplicity. The orthogonal projection onto $\text{Ker } T(0)$ is denoted Π_0 .

Lemma 11.2. [4, Corollary 3.4, Corollary 3.9, Corollary 3.11] *Let T be as above with $I - T'(0)\Pi_0$ invertible. Let $\Omega \subset \mathbb{C} \setminus \{0\}$ be a bounded domain such that $\partial\Omega$ is smooth and transverse to the real axis at each point of $\partial\Omega \cap \mathbb{R}$.*

- (i) *If $\Omega \cap \mathbb{R} = \emptyset$, then for s sufficiently small, $\mathcal{N}(s\Omega) = 0$. So, the characteristic values $z \in \mathcal{Z}(\Omega)$ satisfy $|\Im(z)| = o(|z|)$ near 0.*
- (ii) *Moreover, if $T(0)$ satisfy $\pm T(0) \geq 0$, then the characteristic values z satisfy respectively $\pm \Re(z) \geq 0$ near 0.*

- (iii) For $\delta > 0$ fixed, let $\Gamma^\delta(r, 1) \subset \mathcal{D}$ be defined as in (2.10). Assume that there exists a constant $\gamma > 0$ such that

$$n([r, 1]) = \mathcal{O}(r^{-\gamma}), \quad r \searrow 0,$$

with $n([r, 1])$ growing unboundedly as $r \searrow 0$. Then, there exists a positive sequence $(r_\ell)_\ell$ which tends to 0 such that

$$(11.5) \quad \mathcal{N}(\overline{\Gamma^\delta(r_\ell, 1)}) = n([r_\ell, 1])(1 + o(1)), \quad \ell \rightarrow \infty.$$

- (iv) If we have

$$n([r, 1]) = \Phi(r)(1 + o(1)), \quad r \searrow 0,$$

with $\phi(r(1 \pm \nu)) = \phi(r)(1 + o(1) + \mathcal{O}(\nu))$ for any $\nu > 0$ small enough, then

$$(11.6) \quad \mathcal{N}(\overline{\Gamma^\delta(r, 1)}) = \Phi(r)(1 + o(1)), \quad r \searrow 0.$$

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